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The Classification of Links of Up To And Including Thirteen Crossings

Dylan Joshua Faullin
University of Tennessee - Knoxville

To the Graduate Council:

I am submitting herewith a thesis written by Dylan Joshua Faullin entitled "The Classification of Links of Up To And Including Thirteen Crossings." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

Morwen Thistlethwaite, Major Professor

We have read this thesis and recommend its acceptance:

David Anderson, James Conant

Accepted for the Council:
Dixie L. Thompson

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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The Classification of Links of Up To And Including Thirteen Crossings

A Thesis Presented for the
Master of Science Degree

The University of Tennessee, Knoxville

Dylan Faullin

August 2005

Dedication

While I deeply appreciate the efforts of all whose ultimate drive in life is the acquisition of knowledge from an unrelenting universe, my gratitude currently extends most exceptionally to Dr. Morwen Thistlethwaite. Without his wisdom and guidance, I would not have had the privilege of creating this document or contributing to the interesting field of knot and link theory. I would also like to thank Dr. David Anderson and Dr. James Conant for being on my committee.

I would like to dedicate this work as with all my works big and small, past, present, and future, to my wife Melissa and my beautiful daughter Ayla Grace. It is no small role they play in my life and in motivating me to always try harder. I hope to make the world a better place for them because they ARE my world.

I must also give thanks to my parents, Dan and Sharon, for their support and love. They let me make my own choices and find my own way in the world and for this I am eternally grateful.

I would like to say thanks to the following people or entities in no particular order: Rebecca Faullin, Rachel Faullin, Mike Faullin, Chris Cygan, Joyce Castillo, Jacob Martinez, Aaron Martinez, Gina Martinez, John Martinez, Domingo Montano, Cerdic and Nancy Thompson, Jo Faullin and Penny Fritzen for always celebrating my achievements, my late Great Grandma Yeazel who was a wonderful person who is missed dearly, Mike Lynn Faullin, John Faullin, Corwin Faullin, Chris Donovan, Dominic Rivera and his family, the University of Tennessee at Knoxville, Target Corporation, Albert Einstein, Marvalisa Payne, and all those who have been a part of my life for the past two years.

Because of you all, I am truly blessed.

Abstract

The prime links of up to 13 crossings are classified up to unoriented equivalence.

Table of Contents

Chapter One

Background

Links and their Equivalence Classes

A link of c components is an embedding $S^1 \cup \cdots \cup S^1 \to \mathbb{R}^3$ of c disjoint copies of S^1 into $\mathbb{R}^3$ [2], where S^1 is the unit circle in $\mathbb{R}^2$. Often it is convenient to consider a link as living in $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$. A link with just one component is called a *knot*.

Figure 1 shows two links, one with two components and the other with three components. A *link diagram of a link* L is a projection of L onto the plane, such that the only singular points are transverse double points called *crossings*, together with some way of indicating which of the two strands at each crossing is the overpass. An *alternating link* is a link admitting a diagram where each component passes alternately over and under at successive crossings. Figure 1 shows an alternating link of two components and a non-alternating link of three components.

Links L_1, L_2 are said to be *equivalent* if there exists an ambient isotopy of $\mathbb{S}^3$ taking L_1 to L_2, *i.e.* if there exists a continuous family of homeomorphisms $f_t : \mathbb{S}^3 \to \mathbb{S}^3$ $(0 \leq t \leq 1)$, such that f_0 is the identity map and f_1 maps L_1 onto L_2.

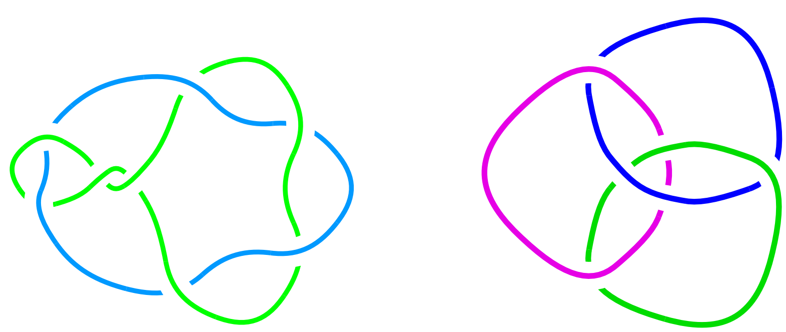


Figure 1: A link of 2 components and a link of 3 components

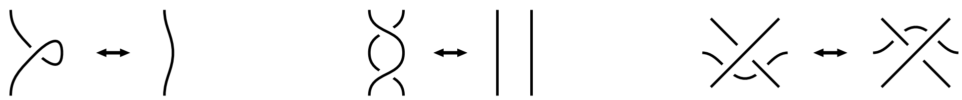


Figure 2: The Reidemeister moves

In practice we consider such an isotopy as a sequence of transformations of link diagrams, called *moves*, beginning with a diagram of L_1 and ending with a diagram of L_2. Any two diagrams of equivalent links are related by a sequence of simple moves called the *Reidemeister moves*[2] (see Figure 2), though in practice we shall need to use more elaborate moves in order to show that certain obstinate pairs of links are equivalent.

The aim of this paper is to classify all prime links of up to 13 crossings, up to *unoriented equivalence*, meaning that in addition to the isotopy equivalence mentioned above we consider that two links are equivalent if they are mirror-images of one another.

It is not possible to show that links L_1, L_2 are distinct just by trying (and failing) to find a sequence of moves relating a diagram of L_1 to one of L_2, since we do not know ahead of time how many moves might be required. Thankfully, other tools have been developed specifically for the purpose of distinguishing links from one another.

A *link invariant* is a property of a link which depends only on its equivalence class. Thus if some link invariant takes distinct values for links L_1, L_2, then the links L_1, L_2 cannot be equivalent.

A simple example of a link invariant is *crossing number*: the crossing number of a link L is the smallest number of crossings of any diagram of L. In practice, however, this invariant is hard to compute, as usually we cannot be certain that a given diagram is minimal. Figure 3 illustrates diagrams of two links that are known to have minimal numbers of crossings; since the first diagram has 6 crossings and the second 7 crossings, these links cannot be equivalent.

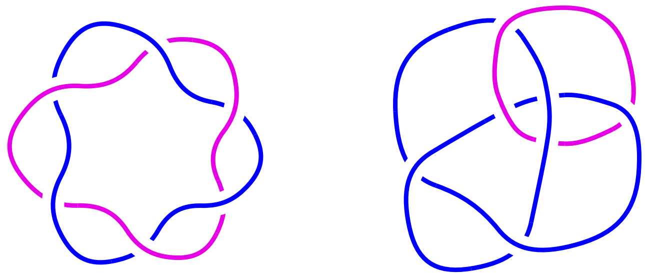


Figure 3: A 6-crossing link and a 7-crossing link

The crossing number is useful as an initial filter, sorting links into sets according to minimal crossing number. However, one should note that just because two links have the same crossing number, it does not mean they are necessarily equivalent. This is true of any general link invariant: if an invariant gives the same value for two links, further scrutiny is needed to determine whether the two links are truly equivalent.

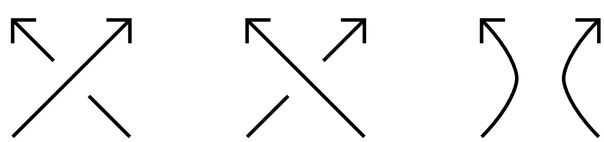
The Jones and HOMFLY Polynomials

An important link invariant, used quite heavily in this paper, is the *Jones polynomial*, discovered by V.F.R. Jones in 1984 [8]. The Jones polynomial $V_L(t)$ of a link L is actually a *Laurent polynomial* in a variable usually denoted t , *i.e.* negative powers of the variable t are allowed. The following set of rules characterizes the Jones polynomial, and may be used to calculate it.

(i) The Jones polynomial is a link isotopy invariant.

(ii) The Jones polynomial of the unknot is 1.

(iii) If oriented link diagrams L_+ , L_- , L_0 are identical except in a neighborhood of one crossing, where they differ as shown:



L_+ $\qquad$ L_- $\qquad$ L_0

then

$$t^{-1} \cdot V_{L_+} \; - \; t \cdot V_{L_-} \; + \; (t^{1/2} - t^{-1/2}) \cdot V_{L_0} \; = \; 0 \; .$$

The relation between Jones polynomials of diagrams conforming to L_+ , L_- , L_0 as above is known as a *skein relation*.

After the discovery of the Jones polynomial, many people wondered if it might be a special case of a more powerful invariant. Indeed, shortly after the discovery of the Jones polynomial, eight people simultaneously discovered the HOMFLY polynomial. Its name is derived from six of the co-discoverers: Hoste, Ocneanu, Millett, Freyd, Lickorish, and Yetter (J. Przytycki and P. Traczyk are also credited with the discovery).

The HOMFLY polynomial $P(\ell, m)$ is a Laurent polynomial in the two variables ℓ, m ; its definition is the same as that of the Jones polynomial, except that the skein relation is replaced by

$$\ell \cdot P_{L_+} \;+\; \ell^{-1} \cdot P_{L_-} \;+\; m \cdot P_{L_0} \;=\; 0 \ .$$

Thus the substitution $\ell = it^{-1}$, $m = i(t^{1/2} - t^{-1/2})$ yields the Jones polynomial from the HOMFLY polynomial. A slightly different substitution gives the Alexander polynomial [10].

The proof that the HOMFLY polynomial is a well-defined link invariant is somewhat complicated, though not deep; a fairly lengthy inductive argument [10] shows that it is invariant under the three Reidemeister moves. In principle it is easy to compute the Jones or HOMFLY polynomial of a link using the above definition; the skein relation allows one to express the polynomial of a link in terms of the polynomials of two simpler links. Although computations are elementary, they are usually tedious, and are best done by computer. The HOMFLY and Jones polynomials used for this classification were computed by means of the program of Ewing-Millett [5].

Link invariants have the fundamental property that if they produce different values on two link diagrams, then the links represented by those diagrams cannot be equivalent. Polynomial invariants are useful in this regard as one can decide by inspection whether two polynomials are distinct, whereas it might be very hard to decide geometrically whether two link diagrams represent equivalent links.

The Jones and HOMFLY polynomials are fairly powerful, but quite often they fail in that they produce the same value for inequivalent links. In particular, they always fail to distinguish pairs of links known as *mutants* [10].

Invariants of Hyperbolic Geometry

According to a deep theorem of Thurston [9], a link complement containing no essential torus or annulus [7] admits a *hyperbolic structure*, *i.e.* a complete Riemannian metric of constant negative curvature. By Mostow-Prasad rigidity [7] such a metric is unique up to isometry, so any invariant of the metric is also a topological invariant of the link complement. In particular, the *volume* of a hyperbolic link complement is always finite, and is already a powerful invariant, although like the Jones and HOMFLY polynomials it is defeated by mutants [11]. The ultimate invariant of hyperbolic link complements, however, is the *canonical triangulation* [7]. It is a uniquely defined way of decomposing the complement into ideal tetrahedra, and is readily computable by the program *SnapPea* [13], encoded as a string of characters. Since the link complement is specified by this string, it is a *complete invariant* of the link complement, *i.e.* two such strings are equal *if and only if* the complements are homeomorphic.

It was proved in 1988 [6] that knots are determined by their complements. Therefore the canonical triangulation is a complete invariant of hyperbolic knots; this was one of the techniques used in [7]. On the other hand, it can happen that inequivalent links of more than one component do have homeomorphic complements [12]. Using SnapPea [13] Dr. Thistlethwaite enhanced the canonical triangulation by incorporating peripheral information for each link component, thus producing a complete invariant of hyperbolic links. Therefore classifying the hyperbolic links was routine, and the bulk of the work in this tabulation was in dealing with the small minority of non-hyperbolic links.

Notation for Links

The notation used for link diagrams is the "alphabetic" notation of [7], as adapted by Dowker and Thistlethwaite from the traditional Gauss code [4]. This system of notation is best understood by looking at the example illustrated in Figure 4.

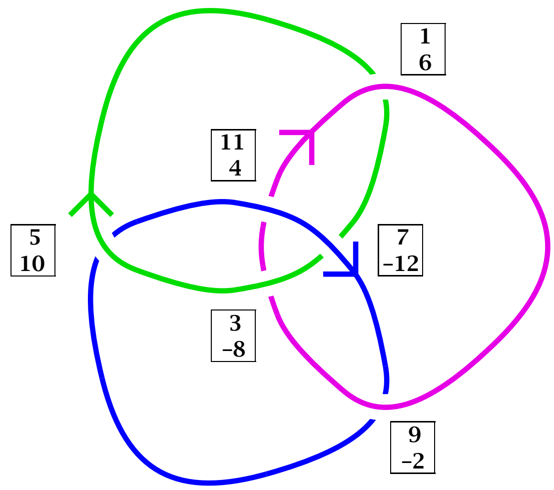


Figure 4: Notation for links

Suppose that we are given an n-crossing diagram of a link L with k components. Each crossing-point of the diagram corresponds to two points on L, one over and the other under. Let P denote the set consisting of these $2n$ points. The idea is to travel around each component of L, numbering successive points in P by the integers from 1 to $2n$. Thus we choose a starting point on some component and travel around that component labelling the successive points in P by $1, 2, 3, \ldots, n_1$, where n_1 is the number of points in P on that component. We then travel around another component, labelling the points $n_1+1, n_1+2, \ldots, n_2$ and so on. For convenience we use the fact that it is always possible to arrange that at each crossing the over- and under-labels have opposite parity, *i.e.* one is even and the other odd. Let us say that each odd integer i corresponds to the even integer a_i. Then each prime diagram is determined by the sequence of even integers $a_1, a_3, \ldots, a_{2n-1}$ [4]. The overcrossing-undercrossing information is given by prefixing a_i by a minus sign if and only if the point labelled i is an undercrossing.

In Figure 4, the right-hand component has nodes labelled $1, 2, 3, 4$, the top left component has nodes $5, 6, 7, 8$, and the lower left component has nodes $9, 10, 11, 12$. Each boxed pair of integers indicates the two nodes at each crossing. Since nodes $3, 7, 9$ are underpasses, the corresponding even integers $8, 12, 2$ are prefixed by minus signs.

It is also necessary to indicate the number of crossings, the number of components and the number of points of P on each component of the link. For the link illustrated in Figure 4, there are six crossings and three components; each component has four nodes.

The coding for a diagram is not unique. There are several choices involved, including ordering of the components, choice of directions and choice of "basepoint" on each component, *i.e.* the node of lowest index.

For efficiency of storage we encode integers by alphabet letters. For the number of crossings and number of components we use the ith lowercase alphabet letter for the integer i, and for the number of nodes on each component and the sequence $a_1, a_3, \ldots, a_{2n-1}$ we use the ith lowercase letter for the integer $2i$, and the ith uppercase letter for the integer $-2i$.

Therefore the complete code for the link diagram in Figure 4 is

$$f \mid c \mid b\,b\,b \mid c\,D\,e\,F\,A\,b \quad ,$$

where f in the first group denotes 6 crossings, c in the second group denotes 3 components, $b\,b\,b$ in the third group denotes 4 nodes on each component and $c\,D\,e\,F\,A\,b$ in the fourth group gives the crossing information.

Chapter Two

The Classification

Introduction

A correct classification of links up to n crossings is a list where each link of up to n crossings is equivalent to precisely one link in the list. There must not be any omissions, nor any duplications. First a "raw" list of links must be obtained, and then the duplicates in that list must be eliminated. Obtaining a raw list is a combinatorial task, achieved by writing a computer program that generates all possibilities. Dr. Thistlethwaite wrote such a program in 2000, which was used to compute a raw list of links up to 13 crossings. The task of eliminating duplicates is more subtle mathematically, and involves the computation of invariants.

As already mentioned in Chapter One, there is a fundamental distinction between hyperbolic links and non-hyperbolic links. The hyperbolic links are classified completely by the canonical triangulation, as computed by SnapPea [13]. Therefore removing the duplicate hyperbolic links was simply a matter of using the UNIX *sort* facility on the character strings denoting canonical triangulations. Removing the non-hyperbolic duplicate links was much more laborious.

Eliminating Composite Links

An important issue that must be dealt with is that of recognizing composite links, as we want our table only to contain prime links. Since the complement of a composite link contains an essential torus, a composite link cannot be hyperbolic. Therefore we need only be concerned with non-hyperbolic links when discussing whether a link is composite. The complement of a link contains a disjoint family (possibly empty) of essential tori that is essentially unique [1], and which splits the complement into elementary pieces. For all the 306 non-hyperbolic links in the final table, it is in fact easy to spot the family of essential tori by inspection; in each case the elementary pieces are very simple, and are known not to constitute a composite link. Thus this issue is resolved simply by generating pictures of the non-hyperbolic links by computer and examining each one. Pictures of all these links are given in Appendix 2. The pictures were generated by a computer program based on the circle-packing software of K. Stephenson [3]

Eliminating the Duplicate Non-hyperbolic Links

The initial list of link codes provided by Dr. Thistlethwaite was generated by a computer program that had already performed several kinds of deformation to test for equivalence [7]. This significantly reduced the number of duplicates to be found by hand, but some undetected duplicates were still in the list.

The detection of these duplicates was done in two stages. First, the Jones polynomial of each non-hyperbolic link was calculated, and the pairs, triples and quadruples of links having the same Jones polynomial were put into a separate file. The HOMFLY polynomials of links not distinguished by the Jones polynomial were then computed. The HOMFLY polynomial managed to distinguish a few of these, but there remained 35 pairs of links that refused to be distinguished. Two of these pairs contained a 13-crossing diagram and a 12-crossing diagram, and all others contained diagrams with equal numbers of crossings. It was suspected that each of these pairs of links were in fact equivalent, but the only way of proving this was to draw them out and perform deformations by hand.

Fortunately it was possible to show in this manner that all 35 pairs were equivalent. The deformations are all illustrated in Appendix 1.

Conclusion

Table 1 below gives the numbers of unoriented prime link types up to 13 crossings, and Table 2 gives the numbers of unoriented non-hyperbolic prime link types up to 13 crossings. The codes for the links are available on request; there are too many hyperbolic links to list here, but as already mentioned the non-hyperbolic links are illustrated in Appendix 2, and their codes are listed in Appendix 3.

Table 1: Number of prime link types by crossing number and number of components

	2	3	4	5	6	7	8	9	10	11	12	13
1	-	1	1	2	3	7	21	49	165	552	2176	9988
2	1	-	1	1	3	8	16	61	185	638	2818	11292
3	-	-	-	-	3	1	10	21	74	329	1095	5731
4	-	-	-	-	-	-	3	1	25	39	307	1161
5	-	-	-	-	-	-	-	-	3	1	47	66
6	-	-	-	-	-	-	-	-	-	-	4	1

Total number of prime link types: 36911

Table 2: Number of non-hyperbolic prime link types by crossing number and number of components

	2	3	4	5	6	7	8	9	10	11	12	13
1	-	1	-	1	-	1	1	1	1	1	-	3
2	1	-	1	-	1	1	1	4	3	9	20	52
3	-	-	-	-	1	-	2	1	4	12	25	93
4	-	-	-	-	-	-	1	-	3	1	19	37
5	-	-	-	-	-	-	-	-	-	-	3	1
6	-	-	-	-	-	-	-	-	-	-	-	-

Total number of prime non-hyperbolic link types: 306

In Table 1 (Table 2), the entry in the column labelled n and row labelled c is the number of link types (non-hyperbolic link types) with n crossings and c components. A dash indicates zero. For example, there are 185 link types with 10 crossings and 2 components, and there are no non-hyperbolic links with 9 crossings and 4 components.

This process completely classified the links up to 13 crossings, and may seem rather arbitrary as there are many ways to proceed in classifying links. However, it comes down to a matter of practicality, and since the above process seemed the easiest to implement it was the one chosen.

The higher the crossing number, the more nonhyperbolic links must be drawn and compared by hand, without the aid of invariants or the computational power of a computer. Given all the useful tools at one's disposal, this is where the real difficulty lies in classifying links today. The number of nonhyperbolic links probably increases exponentially with the number of crossings, so for 14 crossings one would expect there to be over 100 pairs of links requiring individual treatment.

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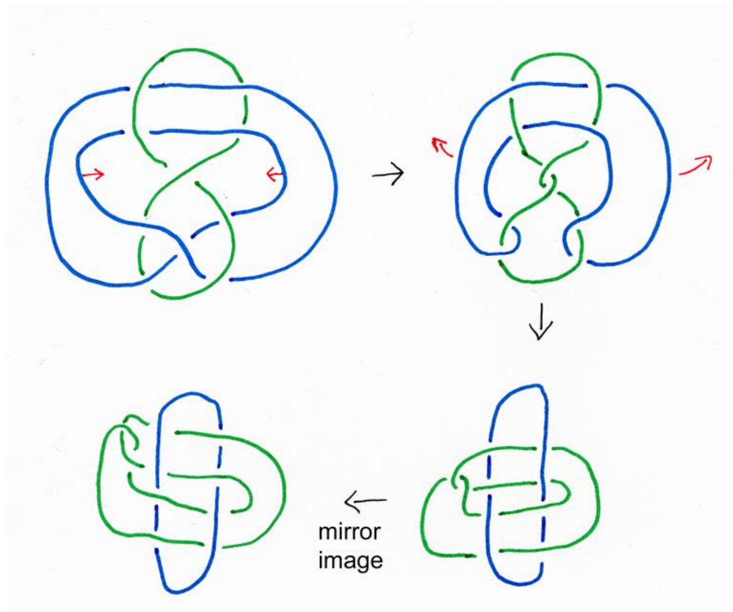
Appendices

Appendix 1: The 35 pairs of equivalent non-hyperbolic links



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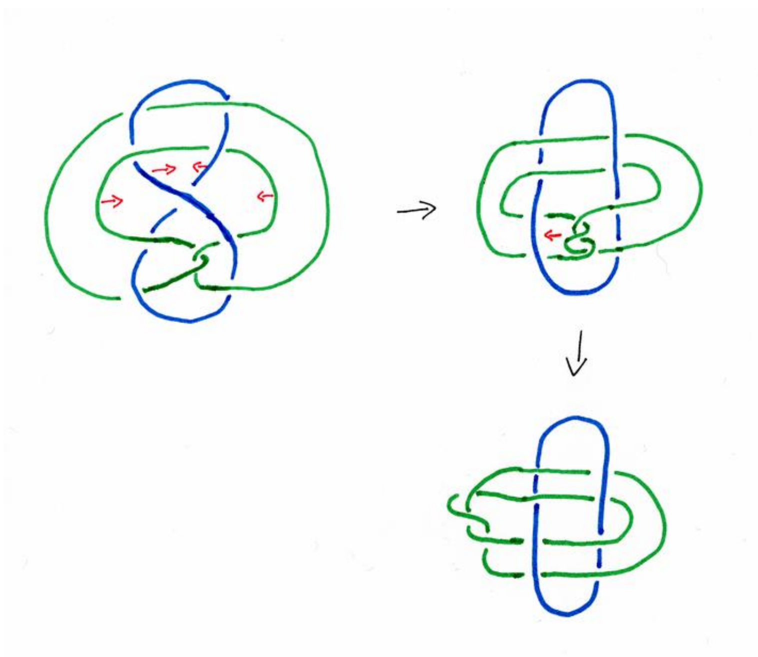

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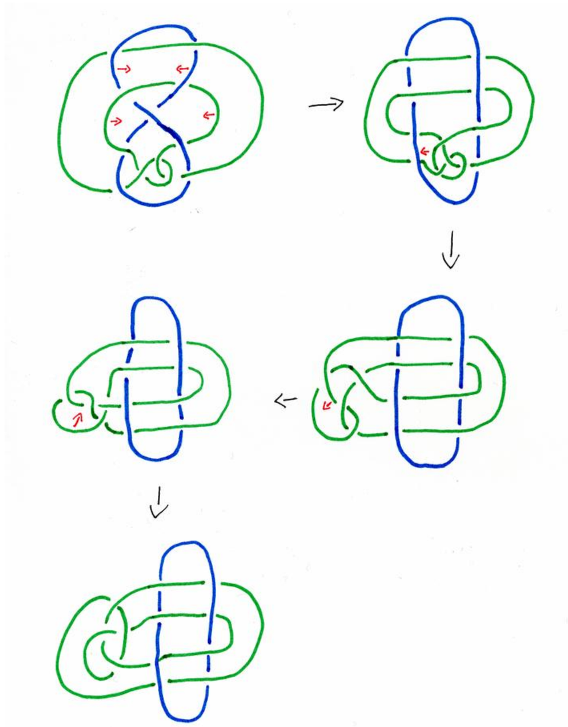

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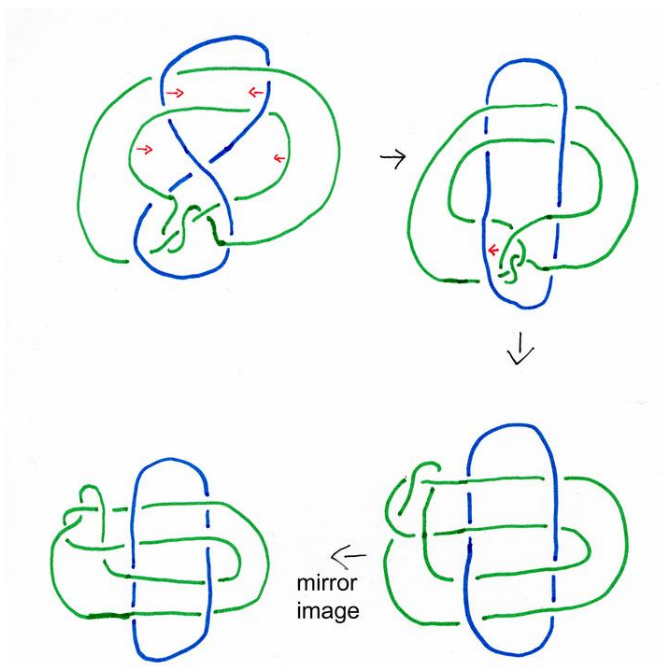

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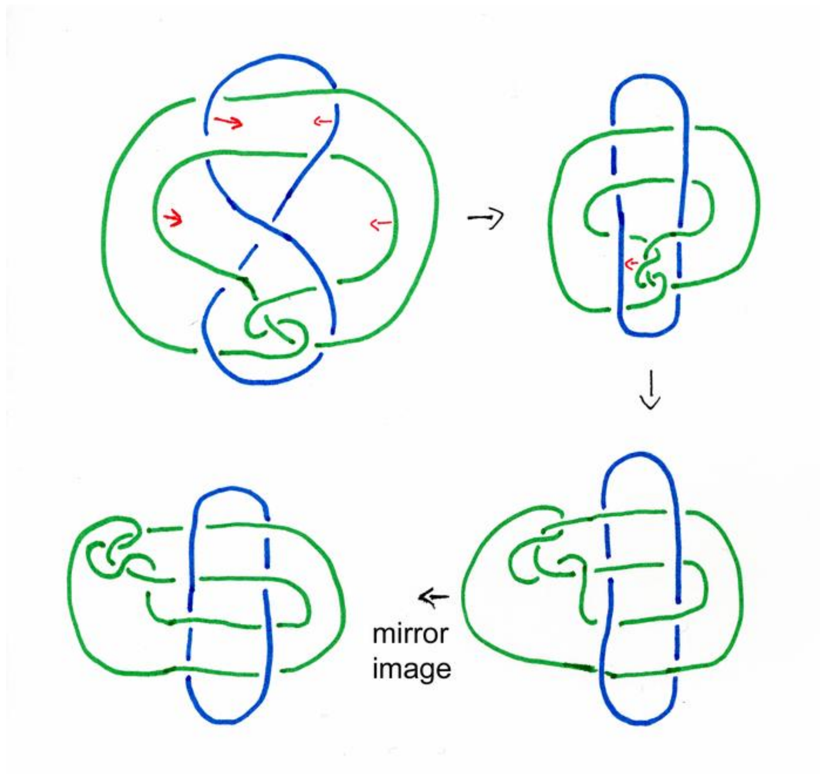

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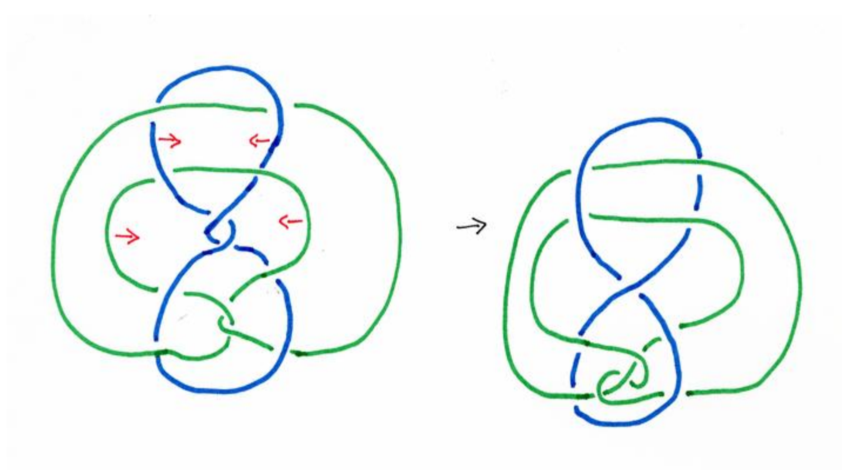

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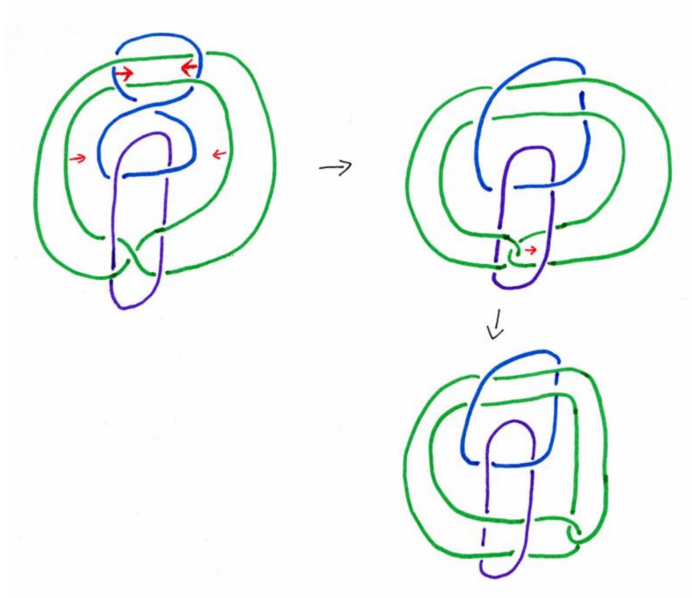

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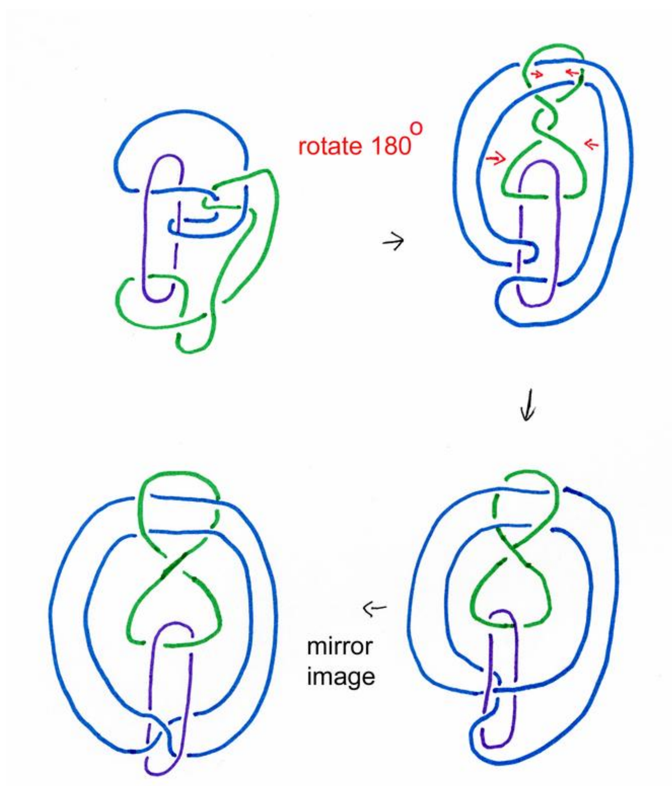

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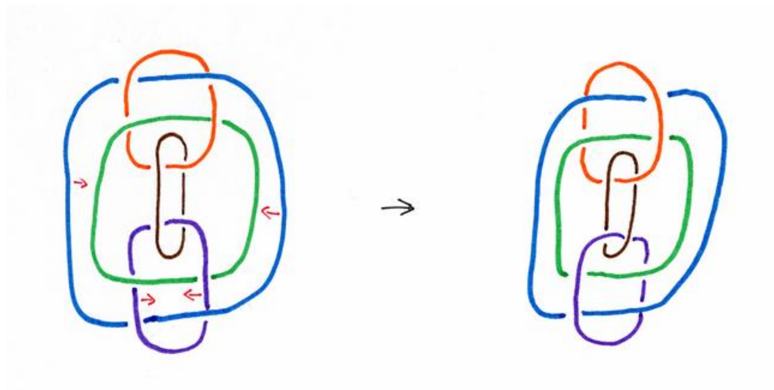

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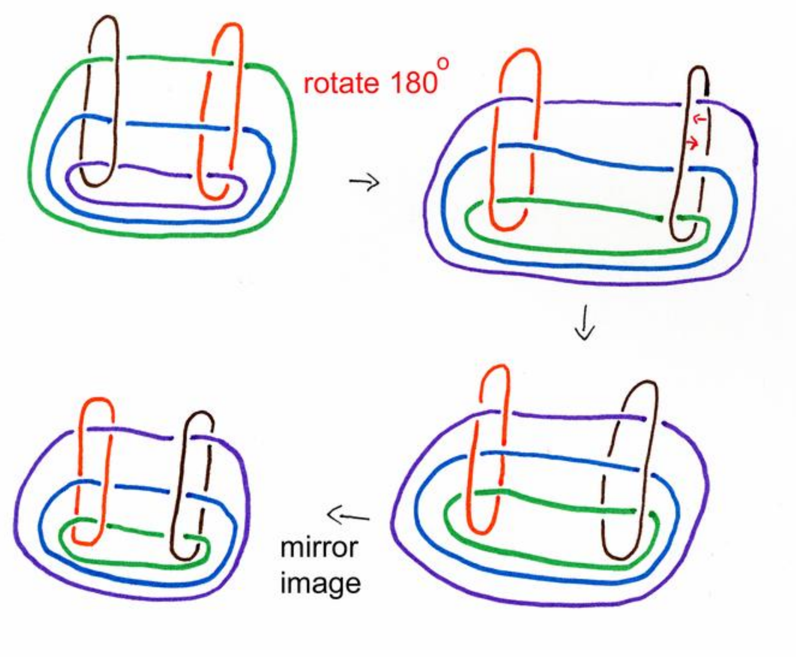

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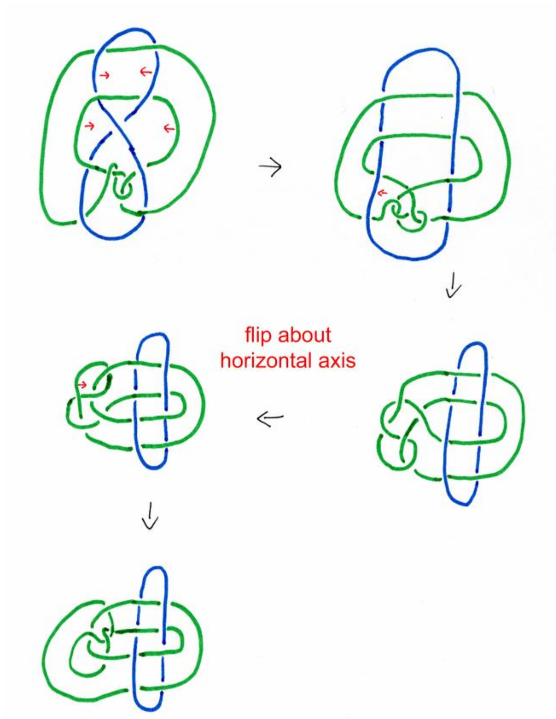

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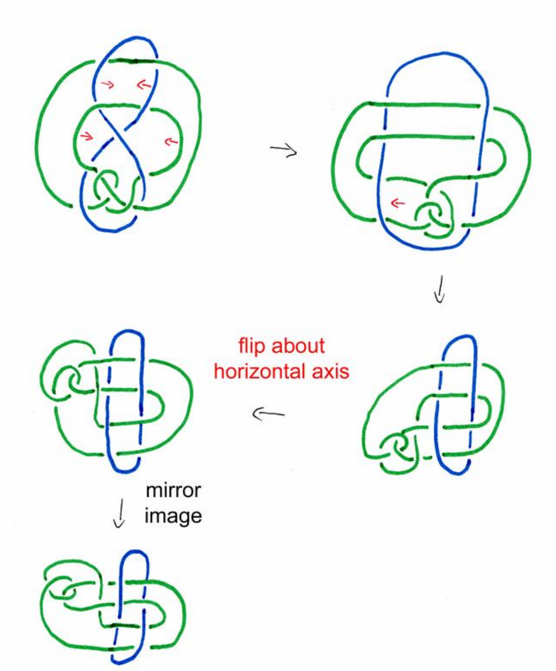

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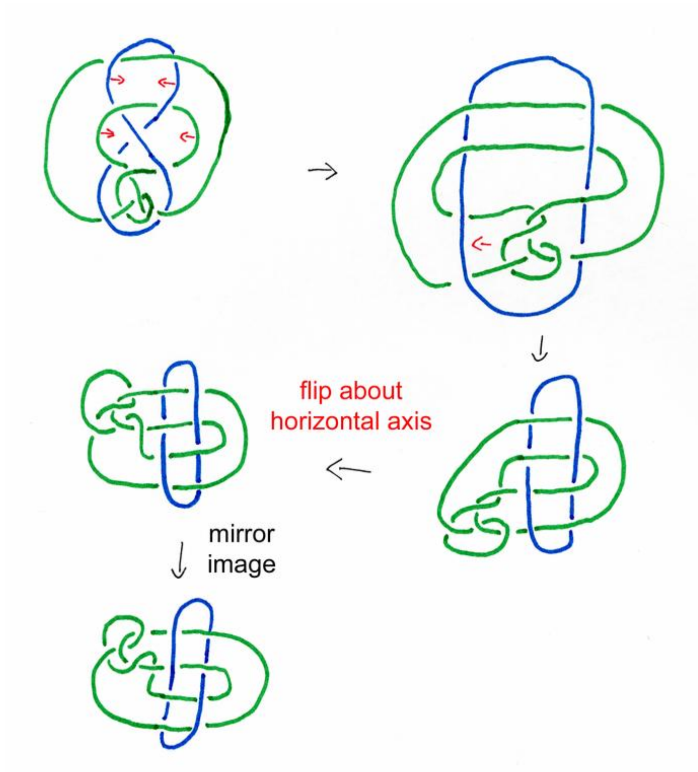

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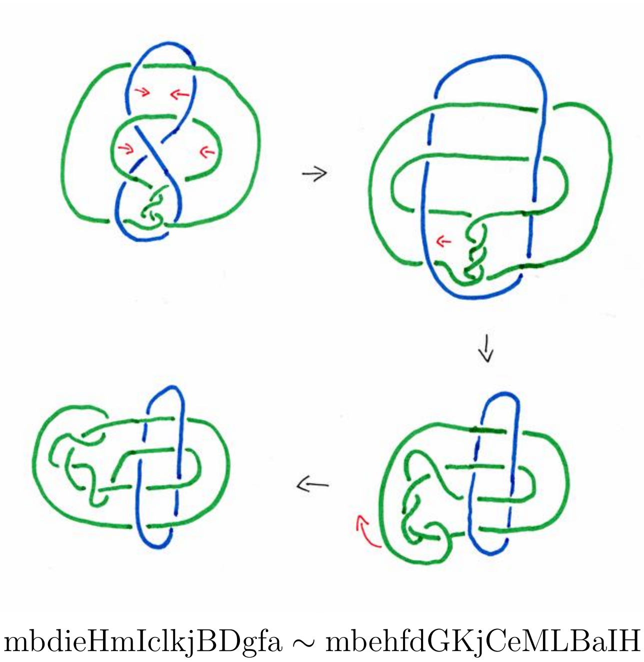

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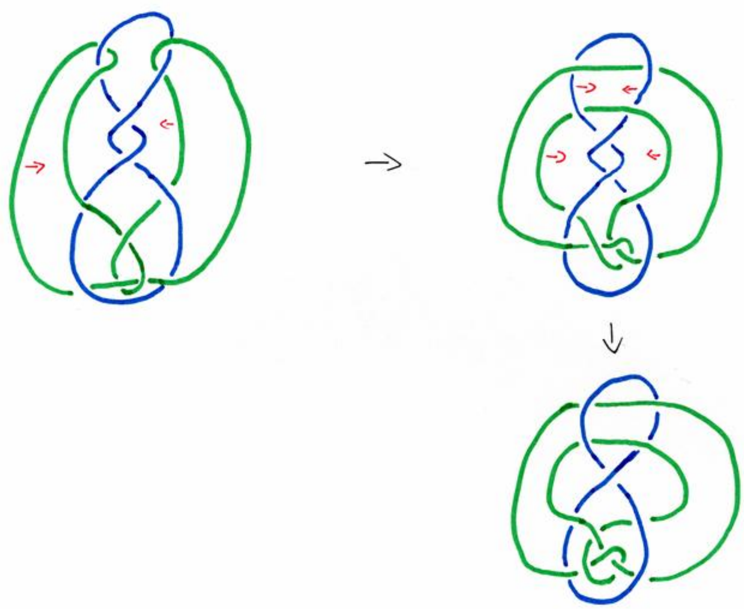
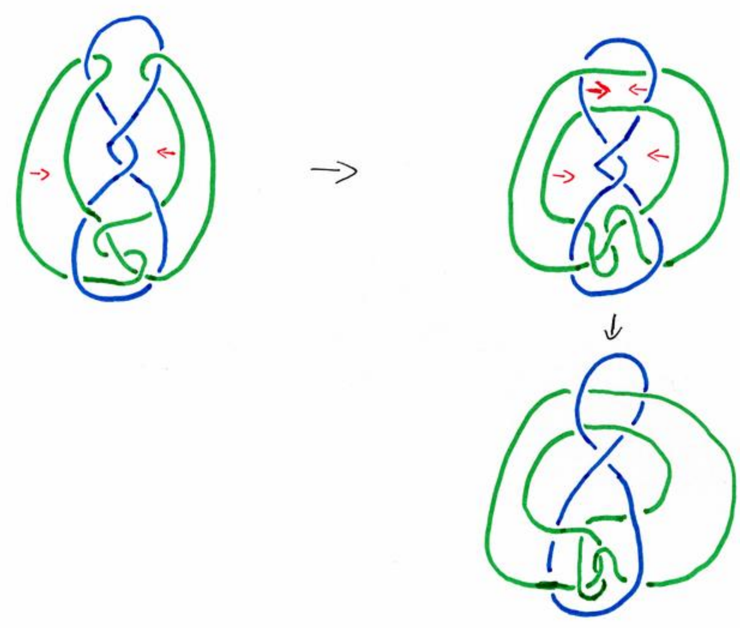
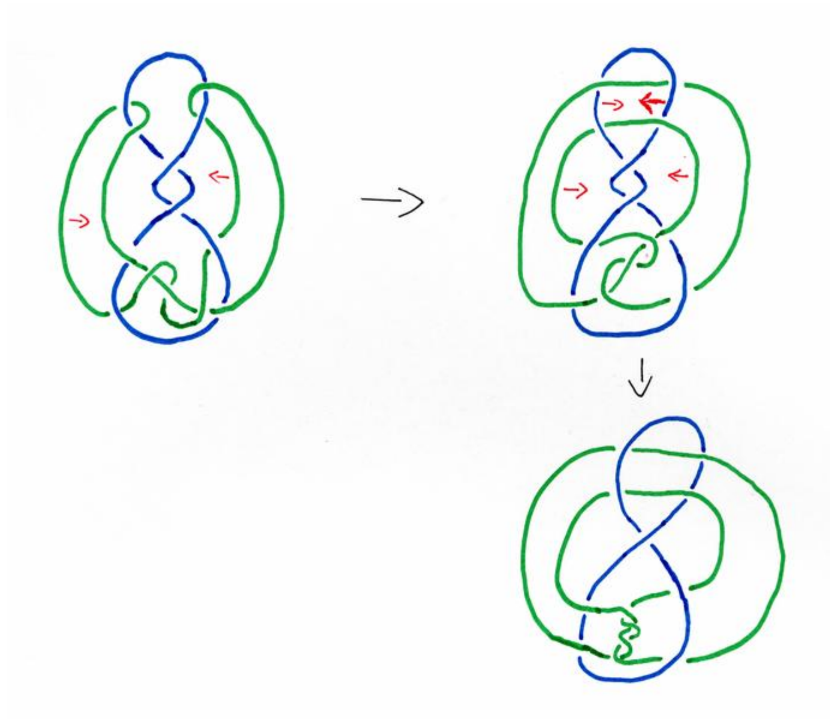


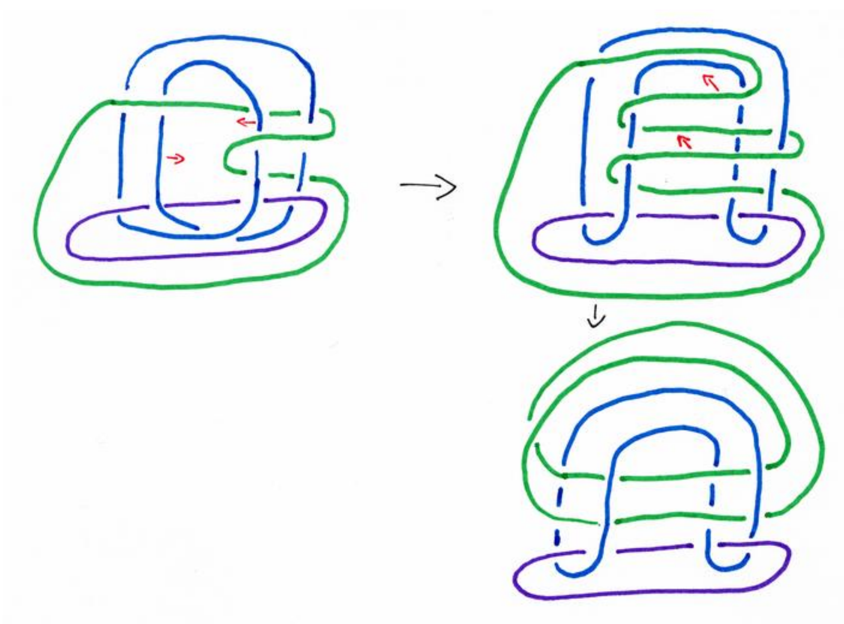
mbehfDgIJcelkABmh $\sim$ mbfggDHKAjBfLCeMI



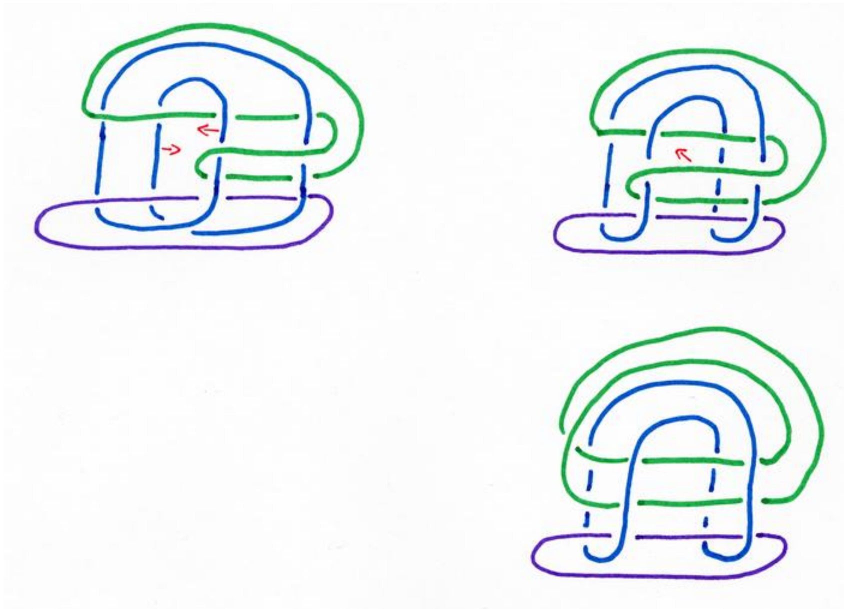
mbehfDgJKecLMBAIH $\sim$ mbfggDHJAkBfLMeCI



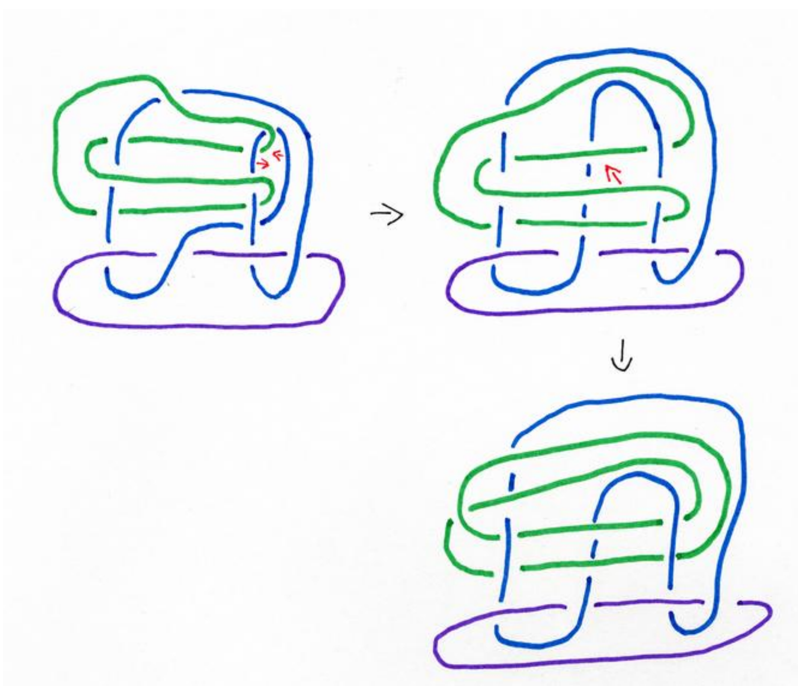
mbehfDgJKecMLBAIH ∼ mbfggDHJAkBfMLeCI



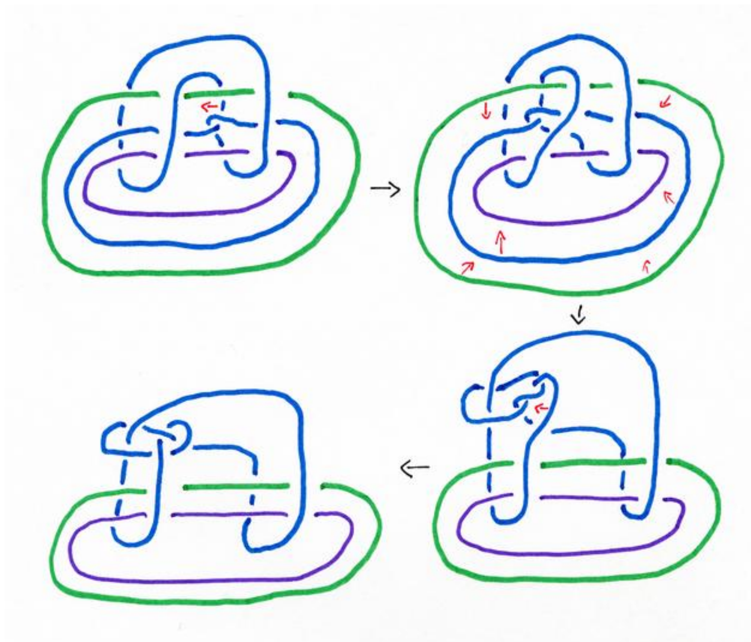
mcbfecFaIjBMkHlgdE ∼ mcbgdcDJgkmaBLfHeI



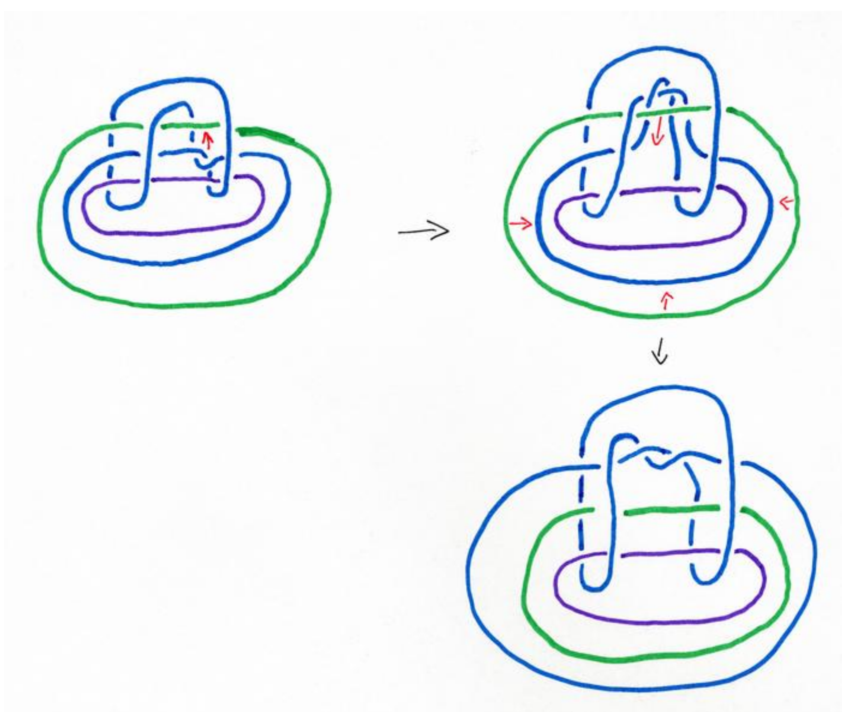
mcbfecfaIjbMkHlgdE $\sim$ mcbgdcDJgkmAbLfHeI



mcbgdcfajlbiKMeGdH $\sim$ mcbfecfaiJbmKhLGDe



mcbibcfaLhbMjdgiEK $\sim$ mcbibcfaHlbIEKGmdj



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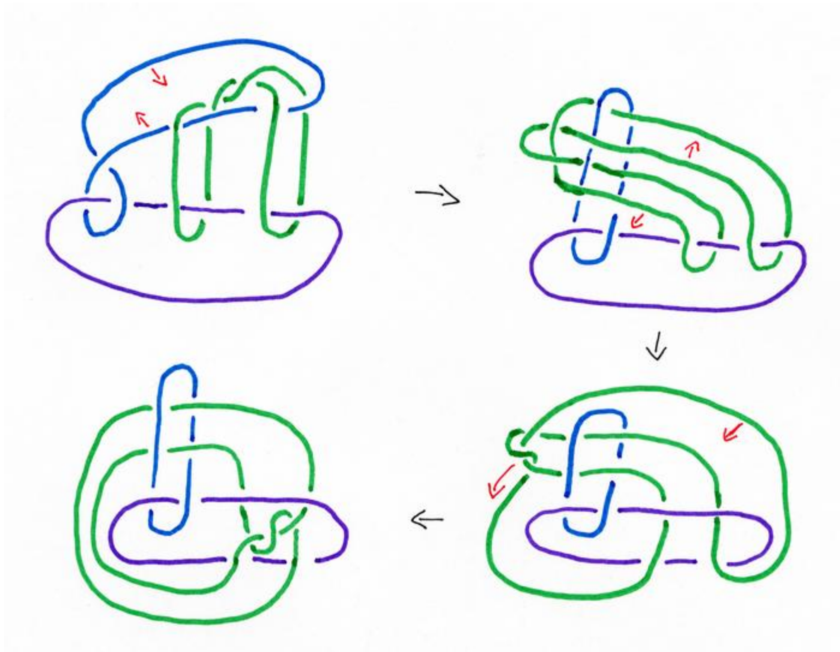
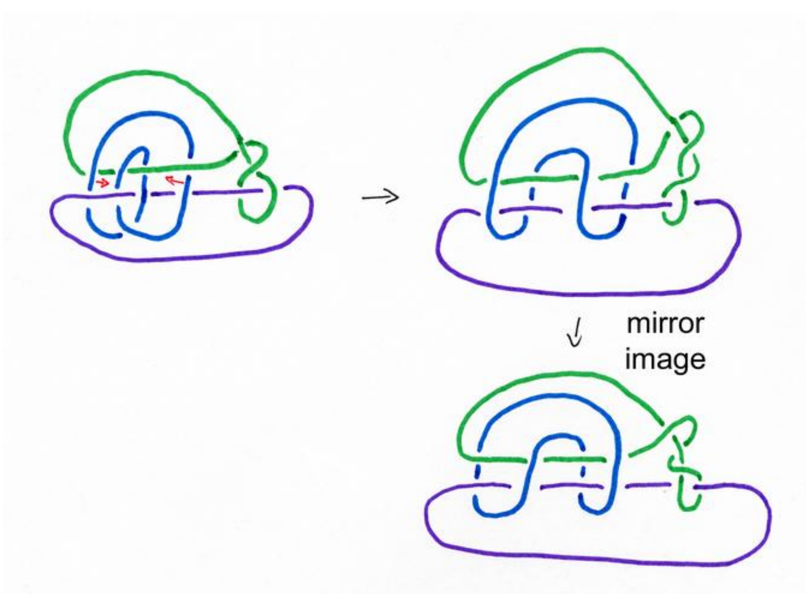
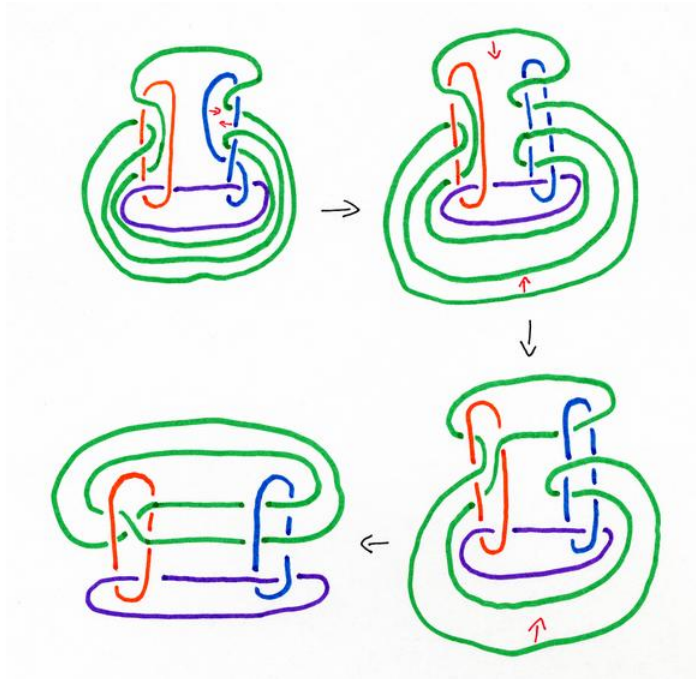


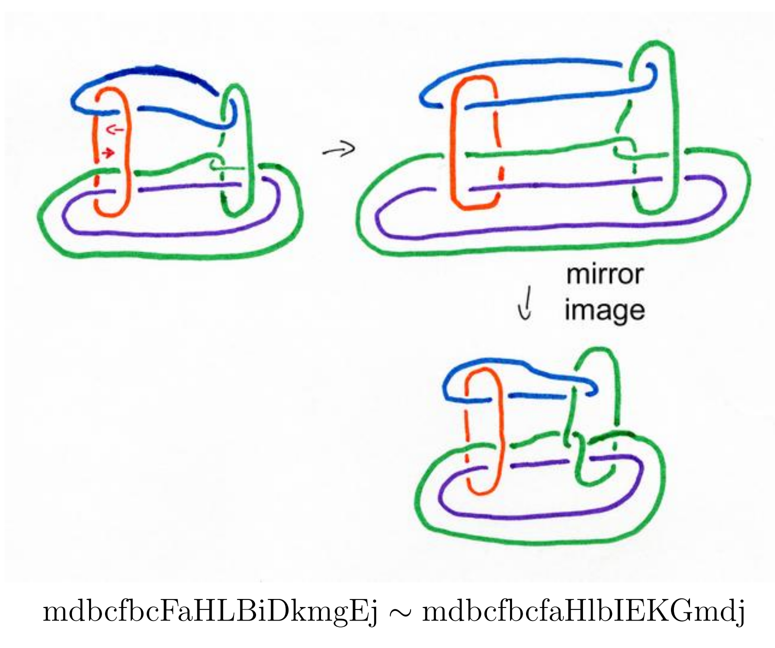
mcccgdgIamJfLKEBHc $\sim$ mccdfdHKagileBmfCj



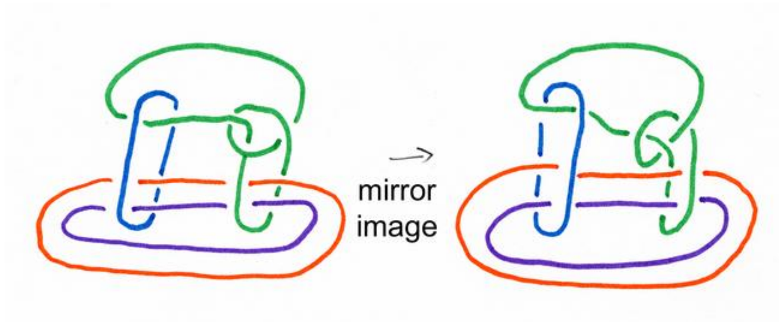
mccdfdfHaJbKCMLGEI $\sim$ mcceedEikGLAbjcmFh



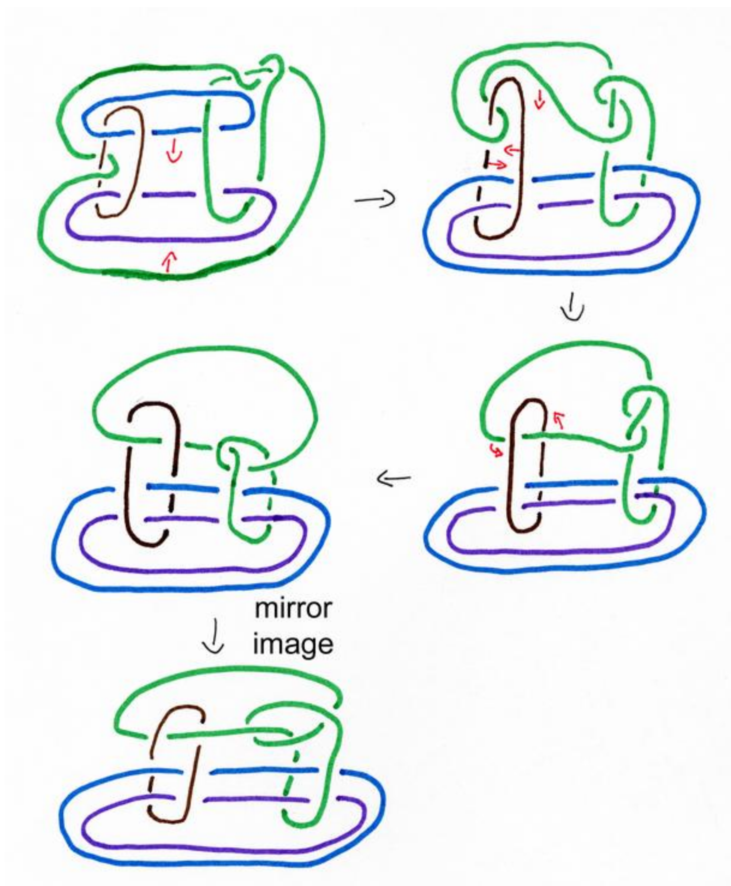
mdbccecfaiJbmKhLGDe ∼ mdbcddcfajlbiKMeGdH



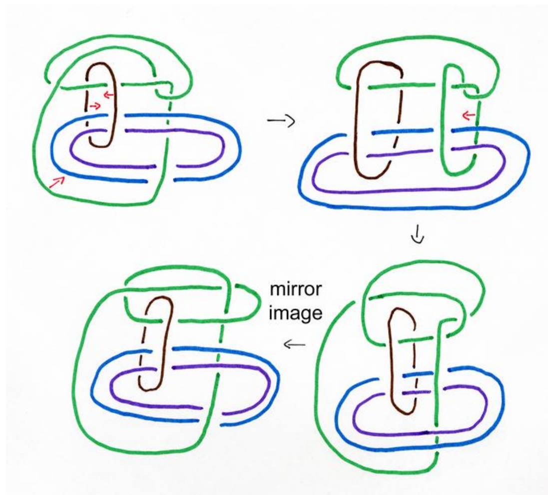
mdbcfbcFaHLBiDkmgEj ∼ mdbcfbcfaHlbIEKGmdj



mdbcfbcfaLhbMJdGIEK ∼ mdbcfbcFaHLmBJDGIEk



mdbcfbcfaLHbMjDgiEK ∼ mdbcfbcFaILBmJKDHEg



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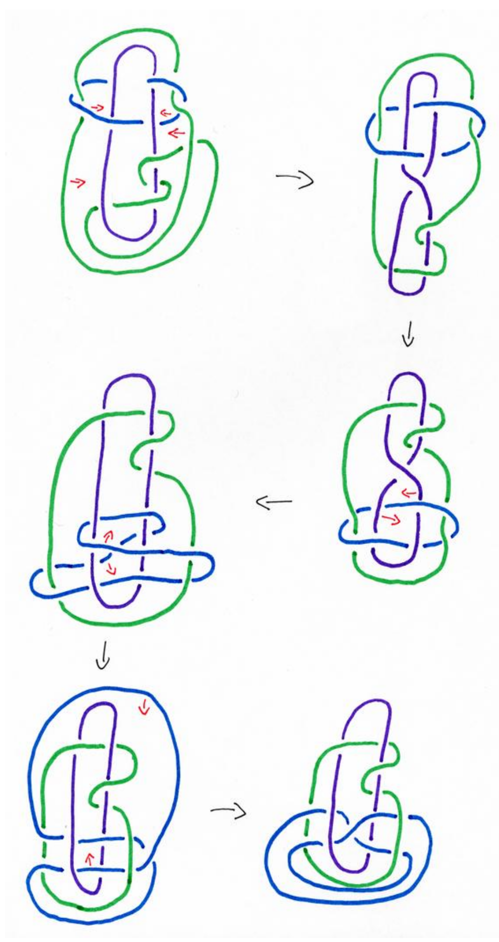
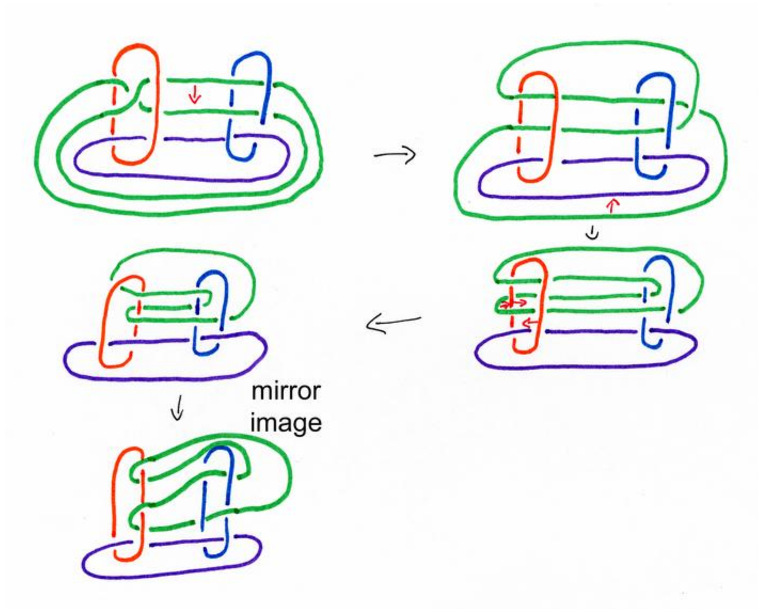
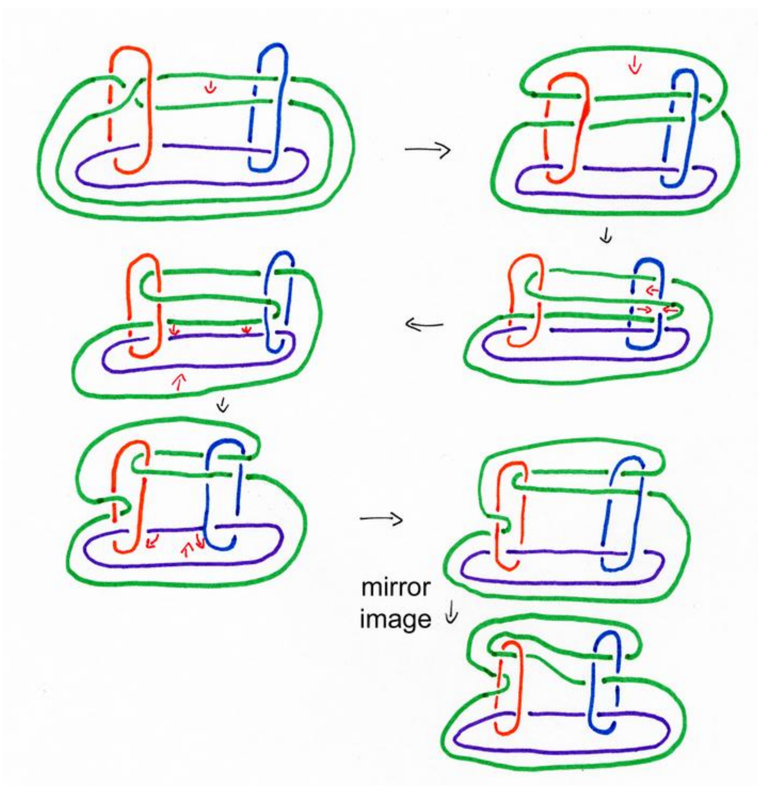


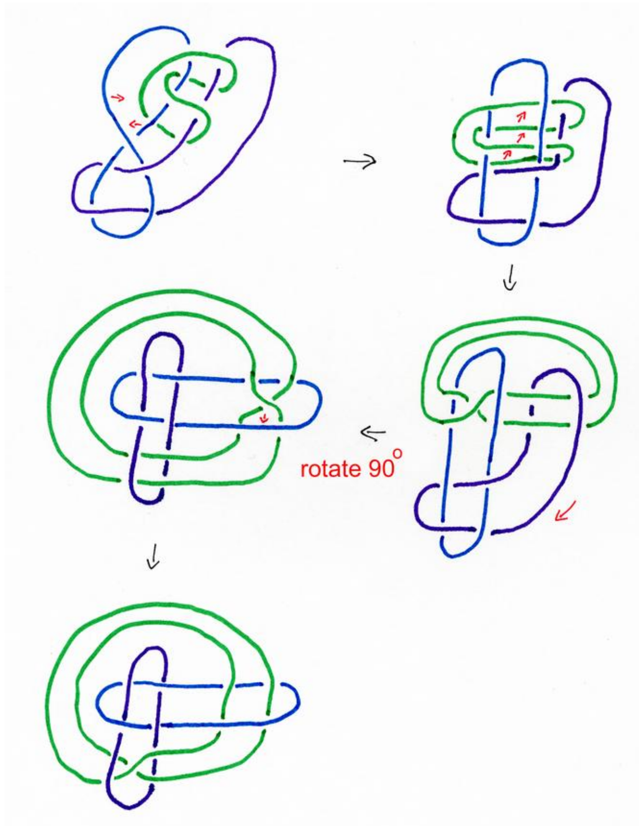
mcddeefIjdakMGLhbC ∼ mcddeegijaLdKbcmHF



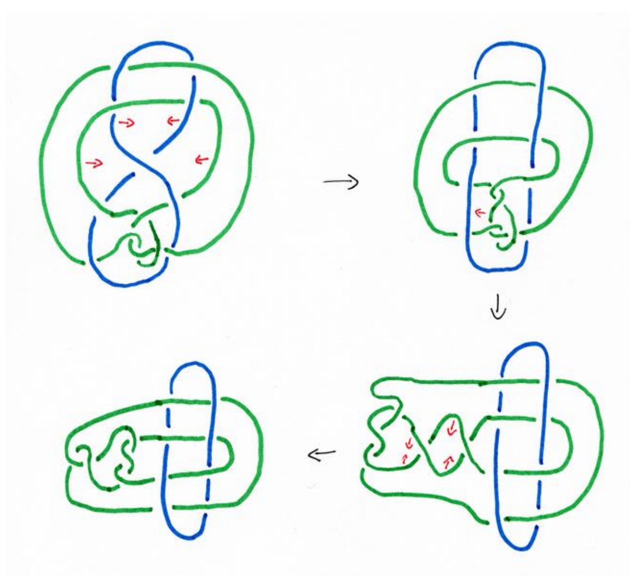
ldbccdcfaIKbjlDhEg ∼ mdbcccecFaIjBMkHlgdE



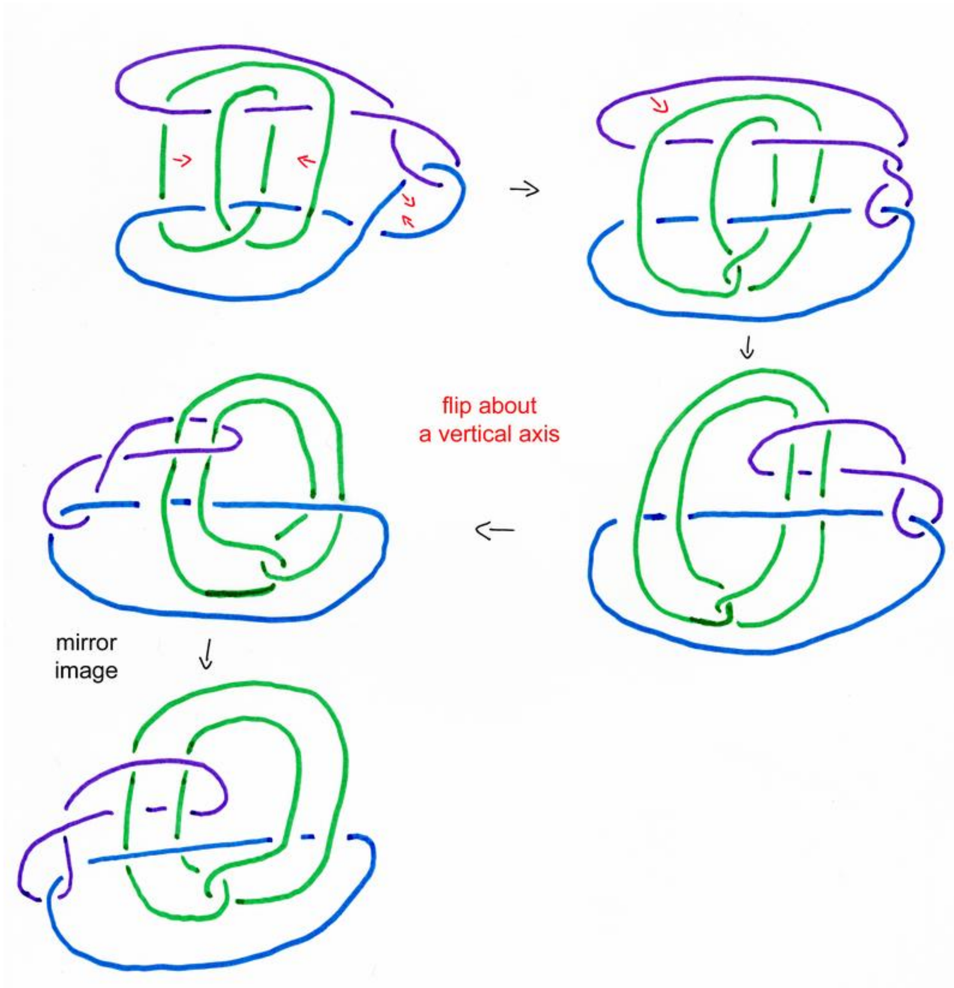
ldbccdcFaIKBjlDhEg ∼ mdbcccecfaIjbMkHlgdE



mcddeefIjadkMGLhbC ∼ mcddeegIJaKbLDCMHF



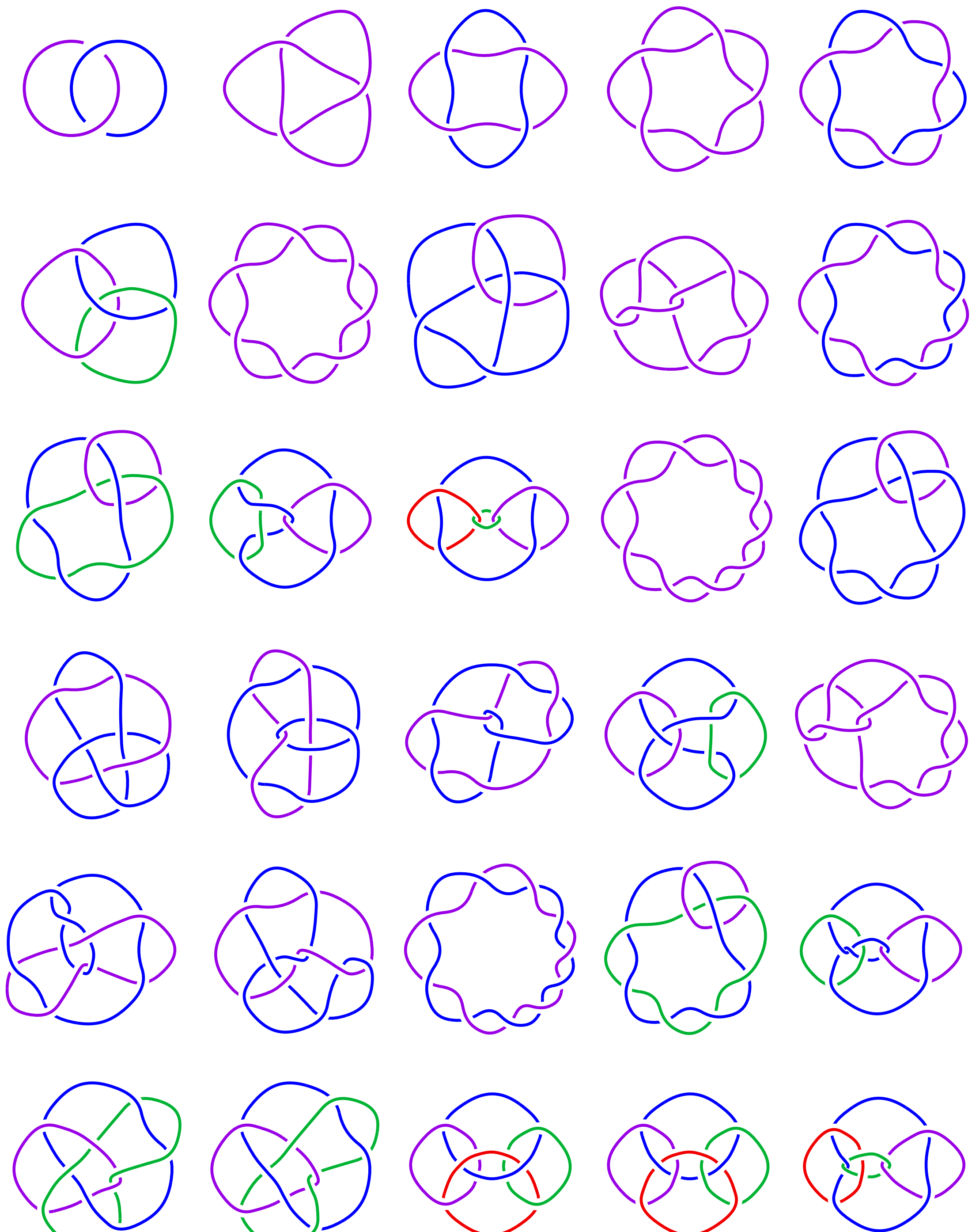
mbehfdGKjCeLMBaIH ∼ mbdieHmIckljBDgfa



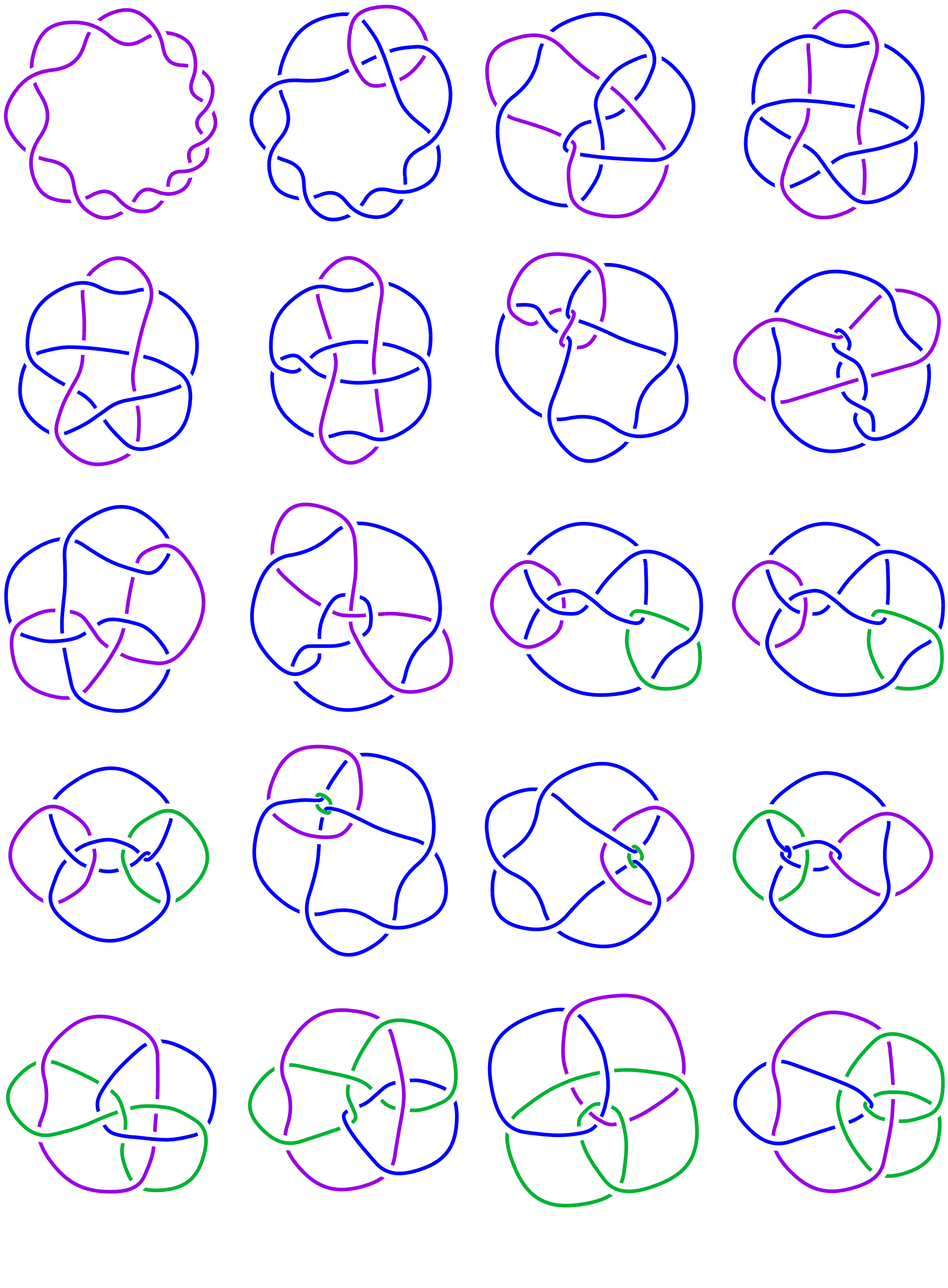
mccdfdhJagKmclBFie $\sim$ mcceedeIKgLabJCMFH

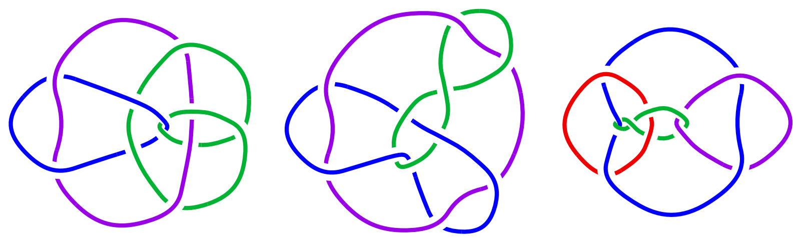
Appendix 2

Non-hyperbolic links from 2 to 10 crossings

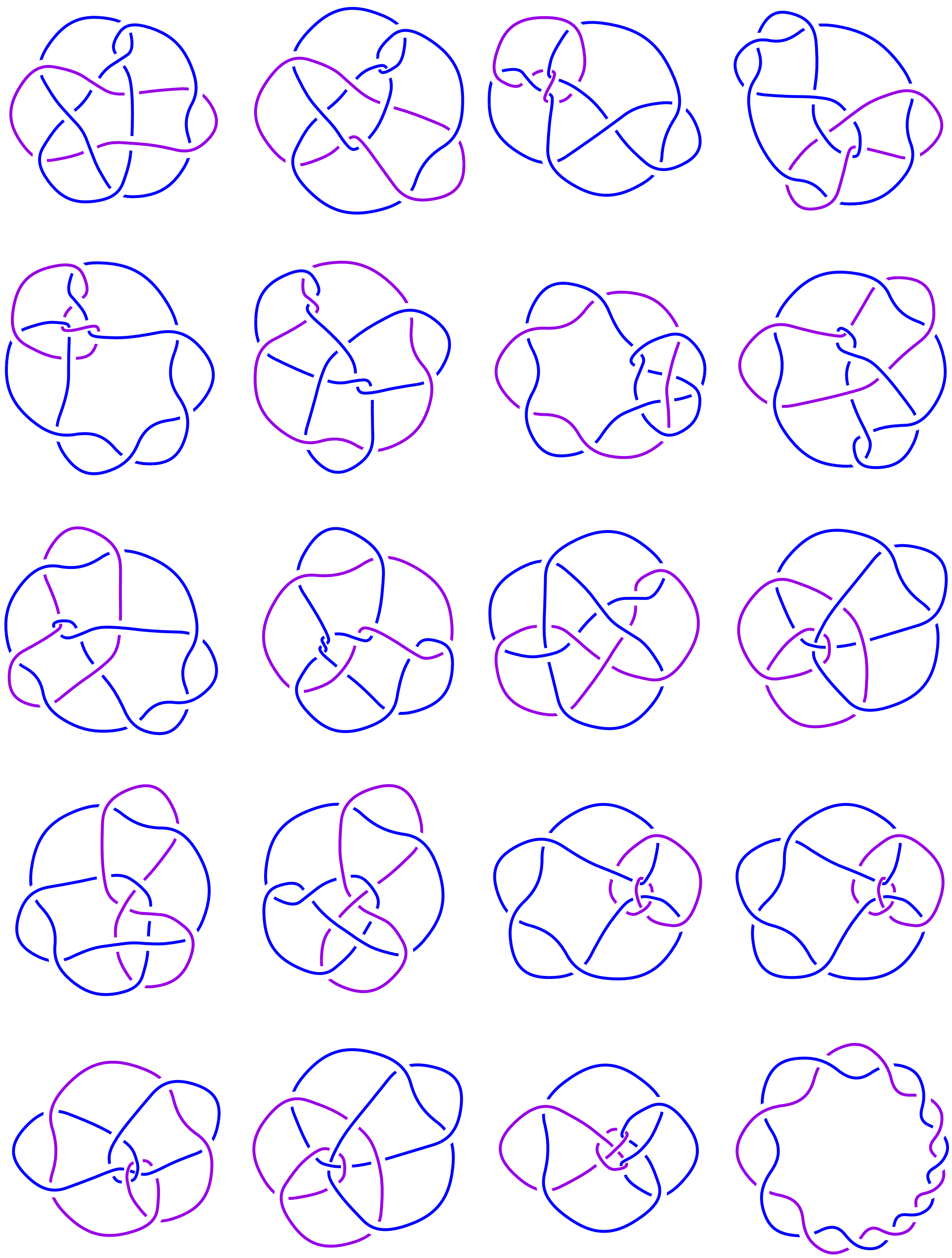


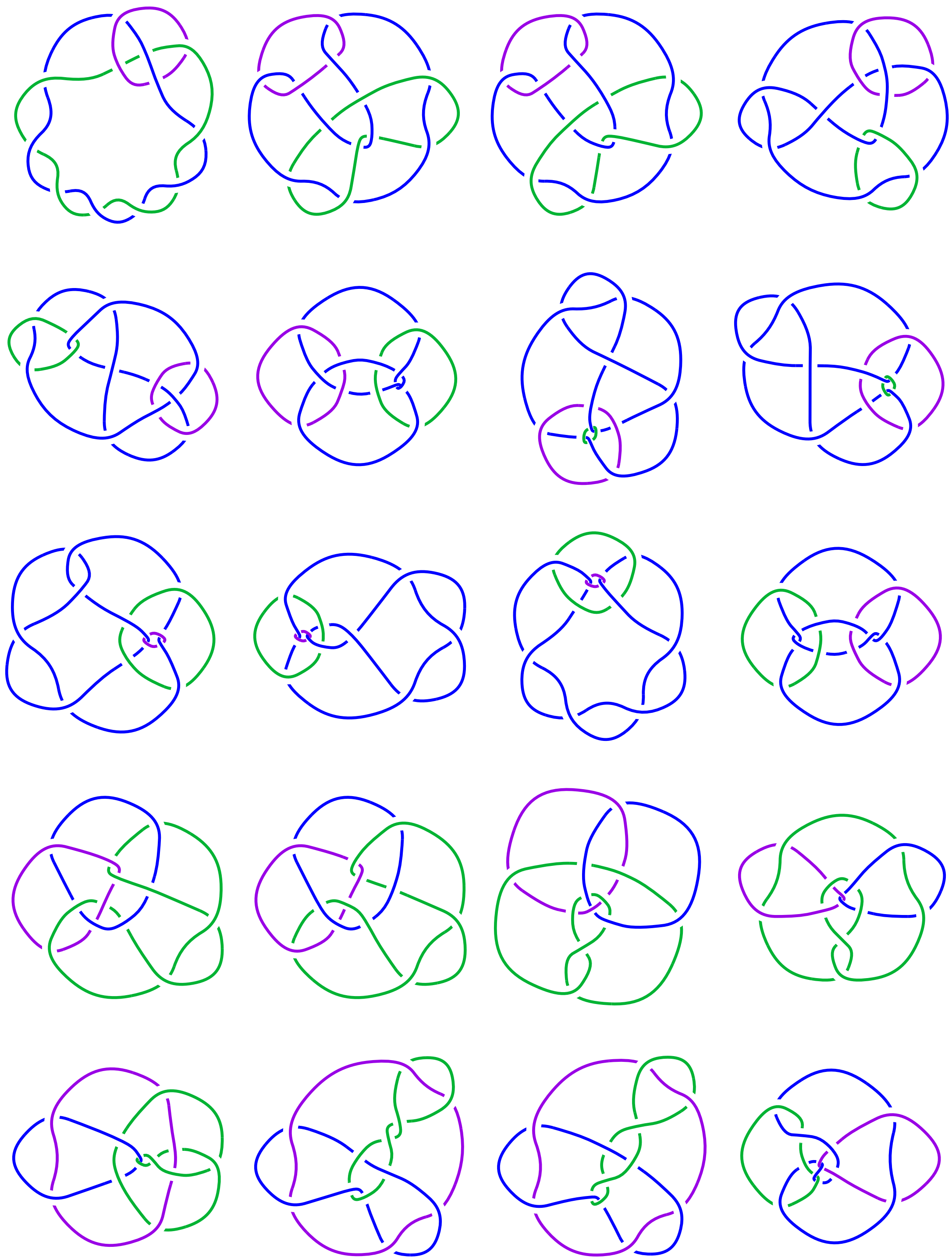
Non-hyperbolic links of 11 crossings

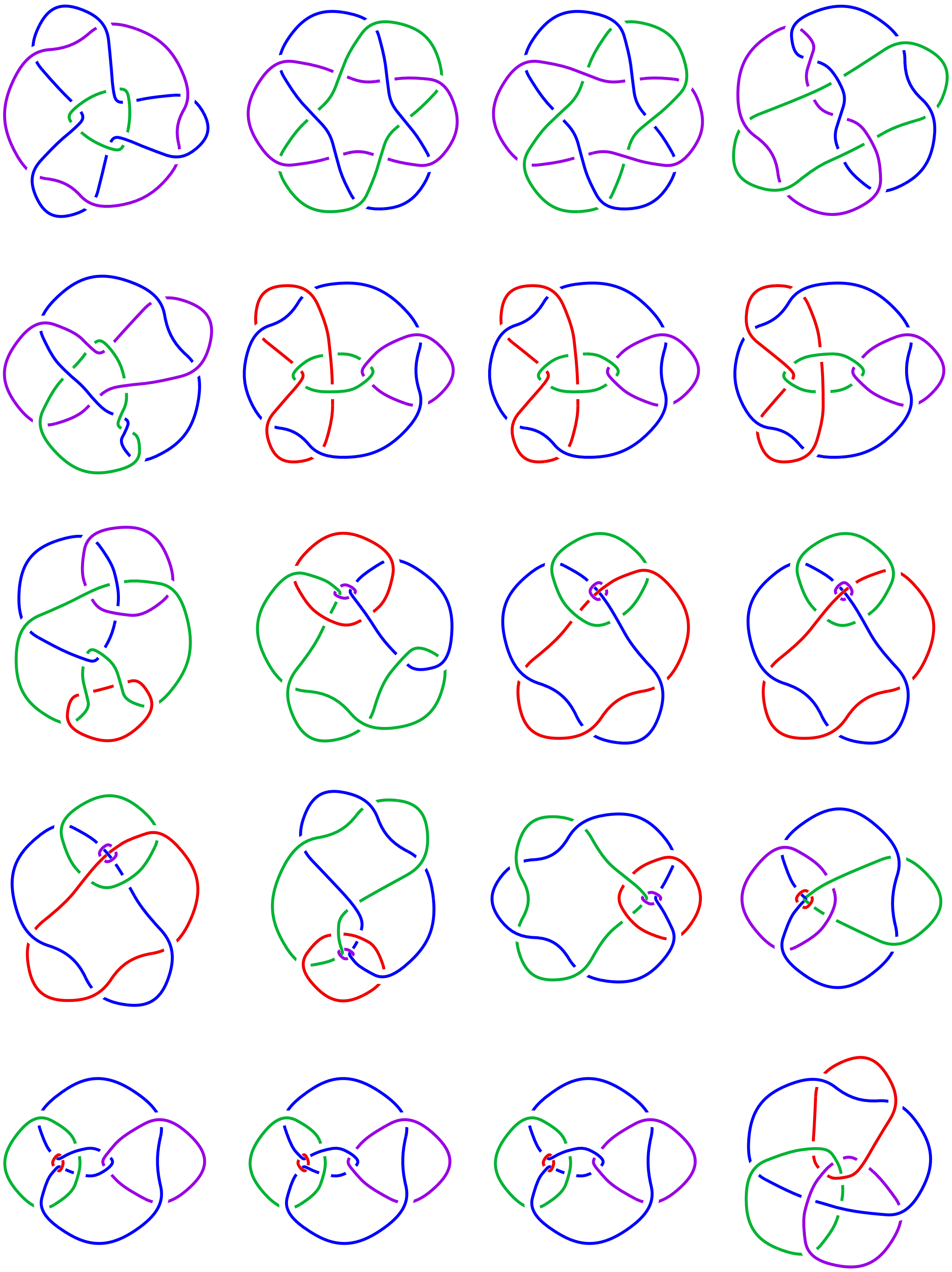


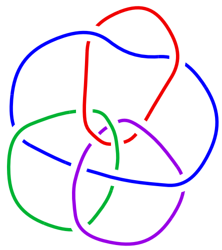
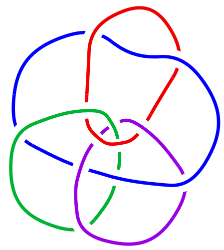
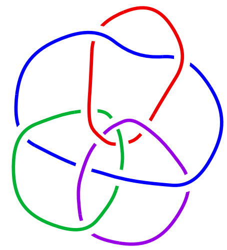
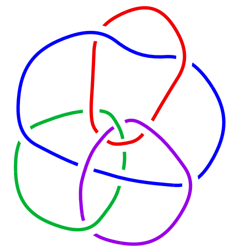


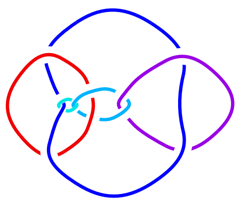
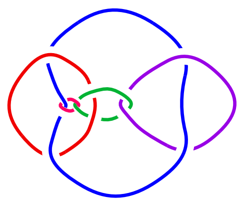
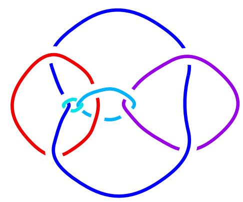
Non-hyperbolic links of 12 crossings



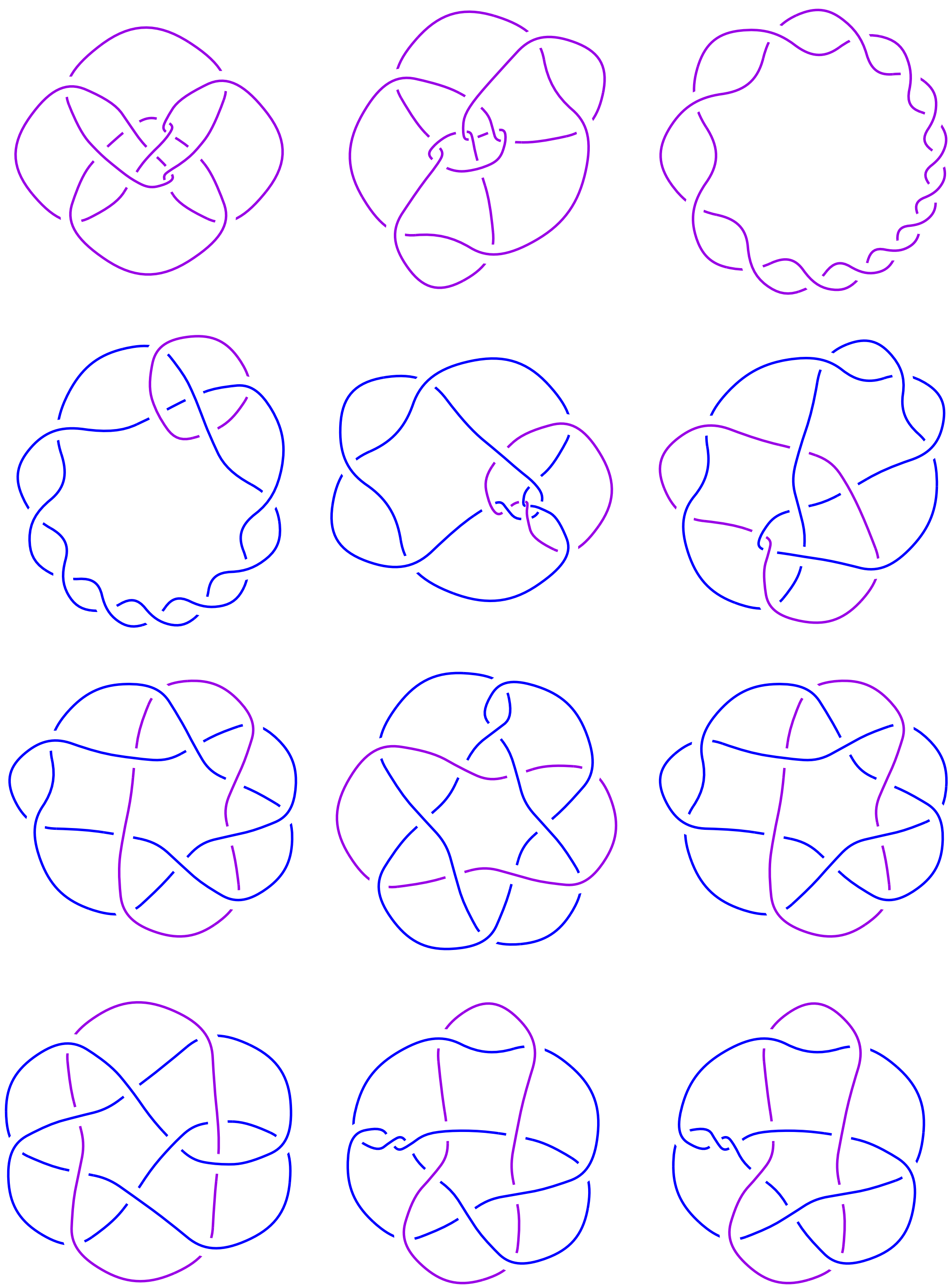


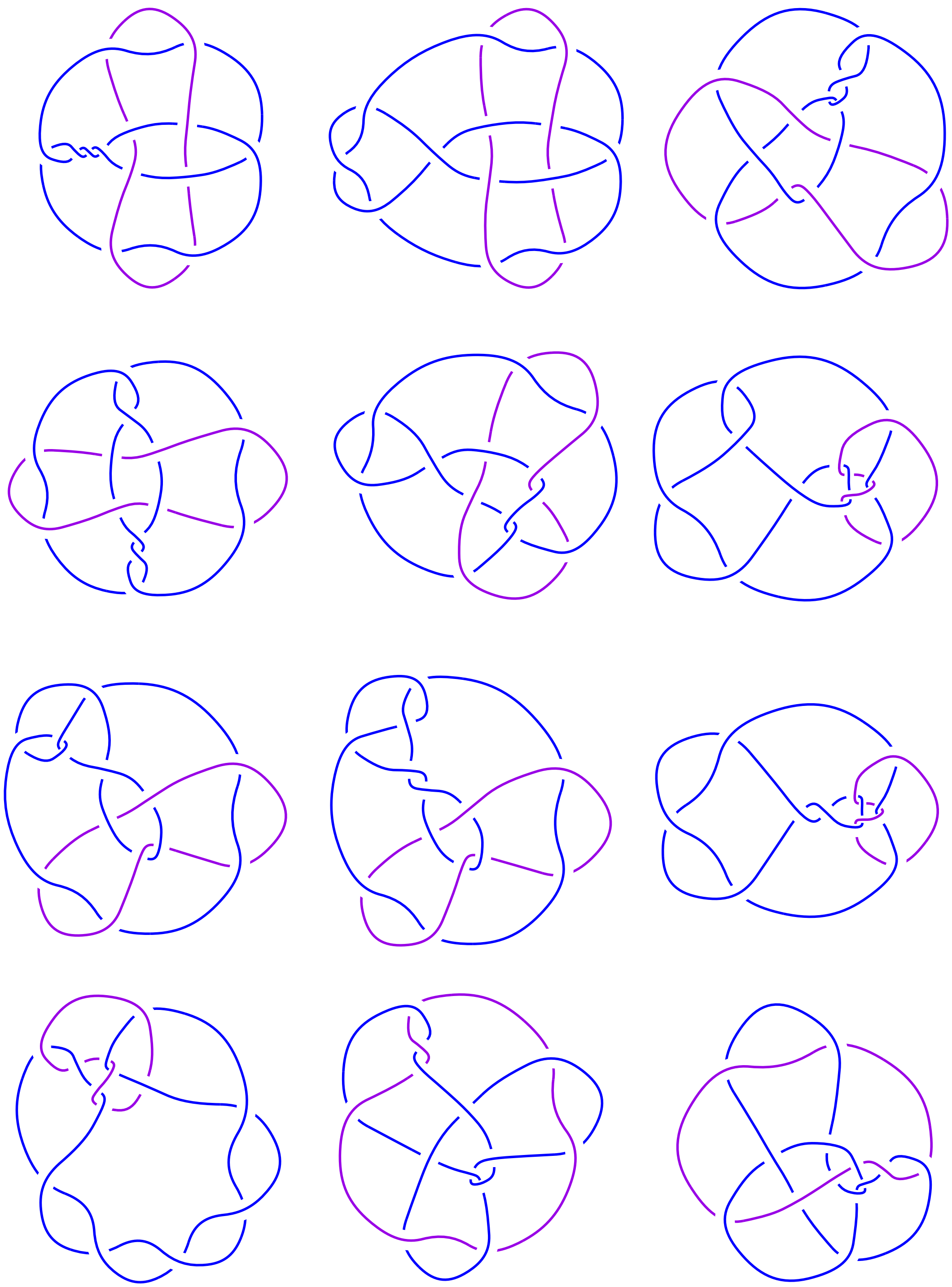


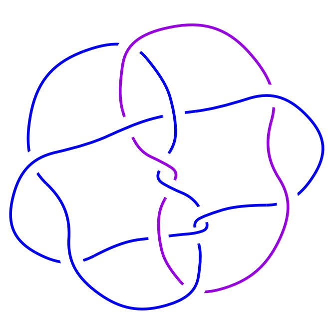
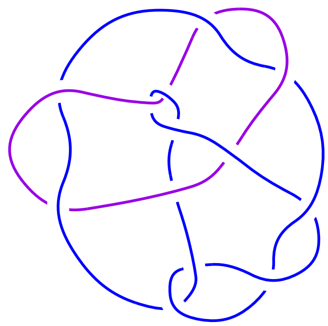
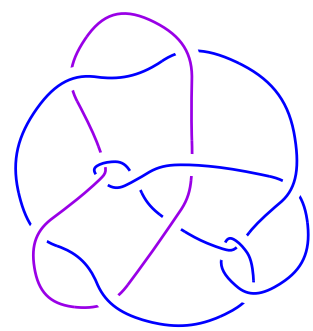
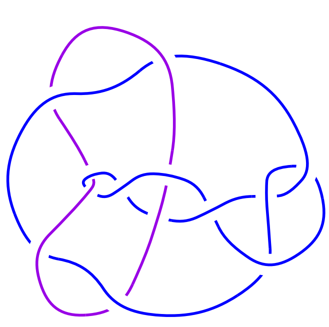
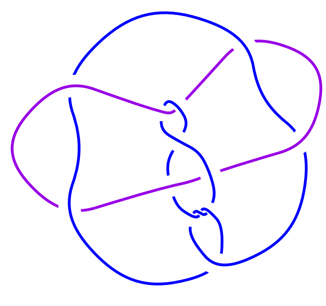
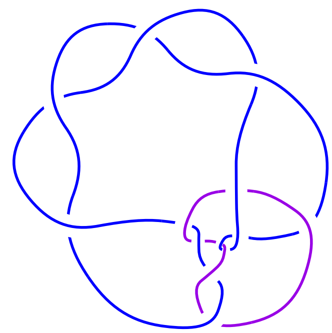
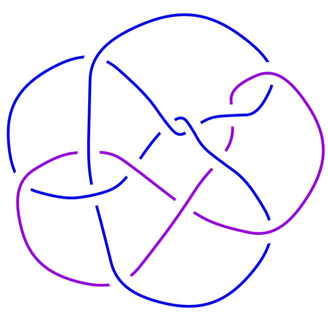
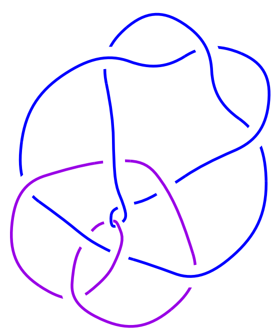
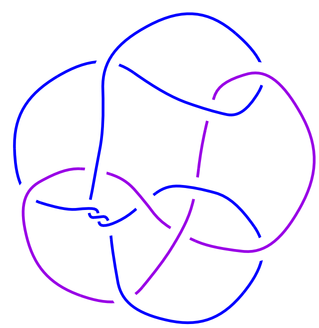
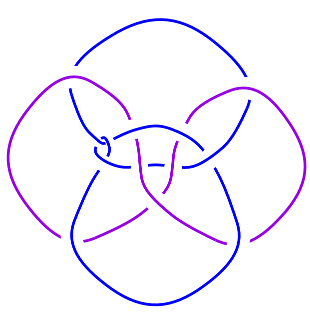
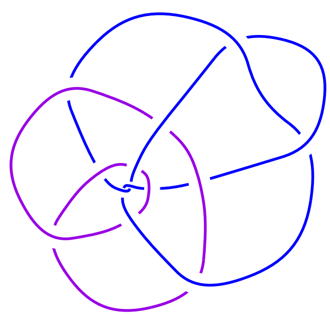
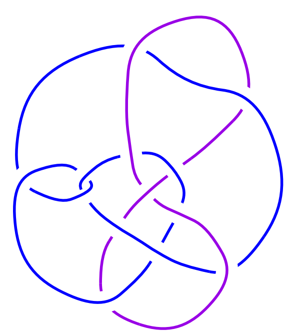





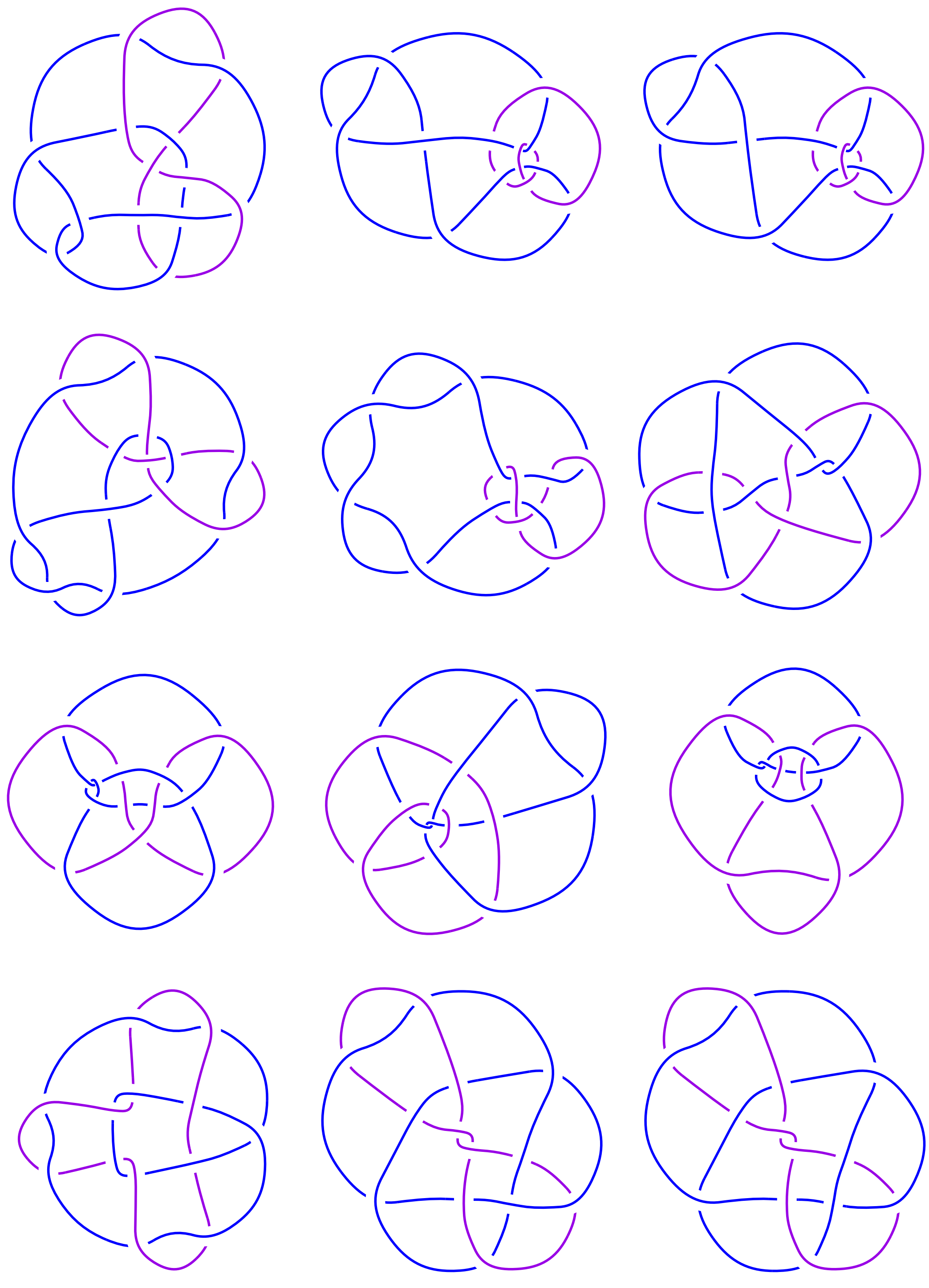




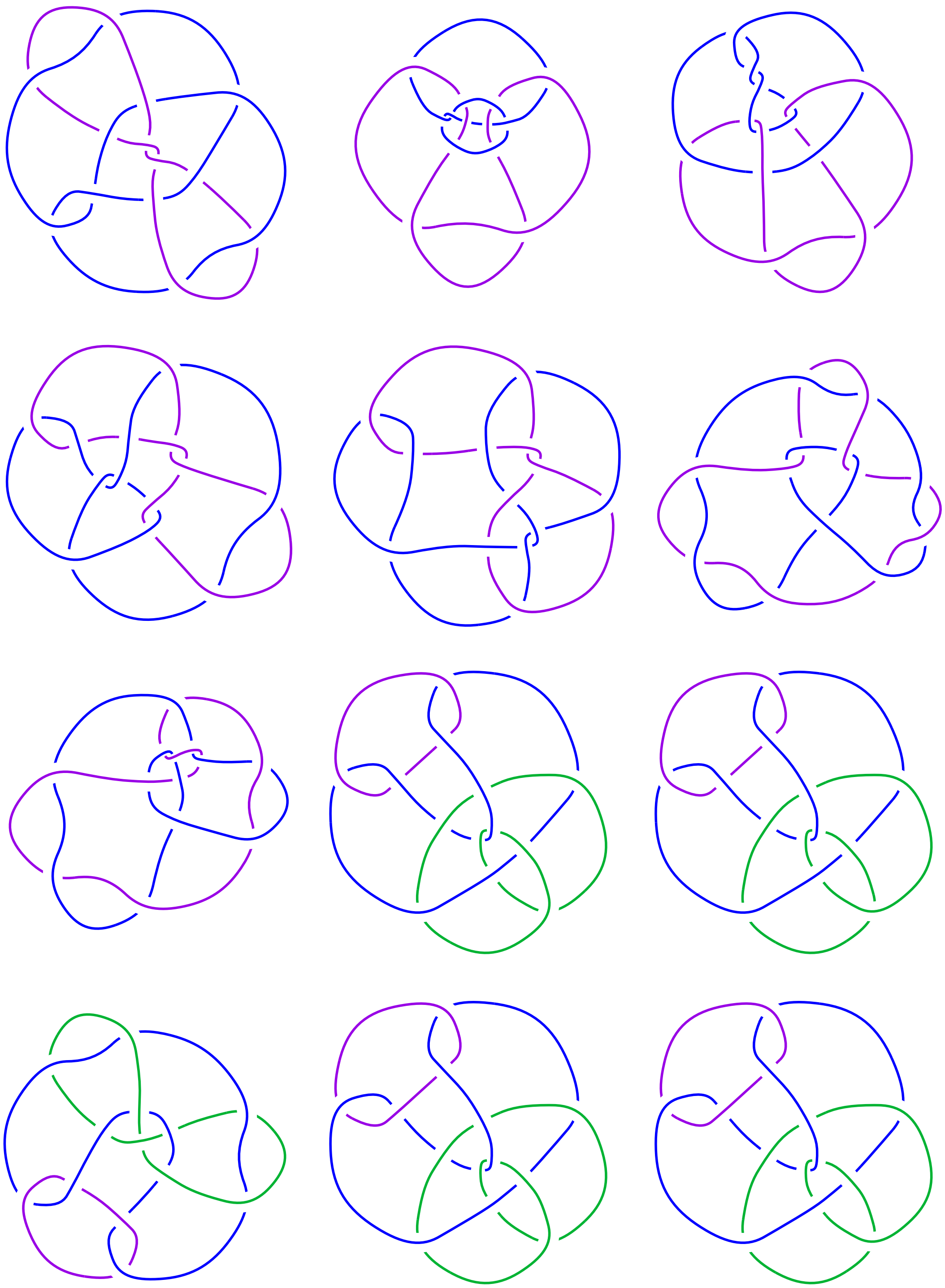
Non-hyperbolic links of 13 crossings

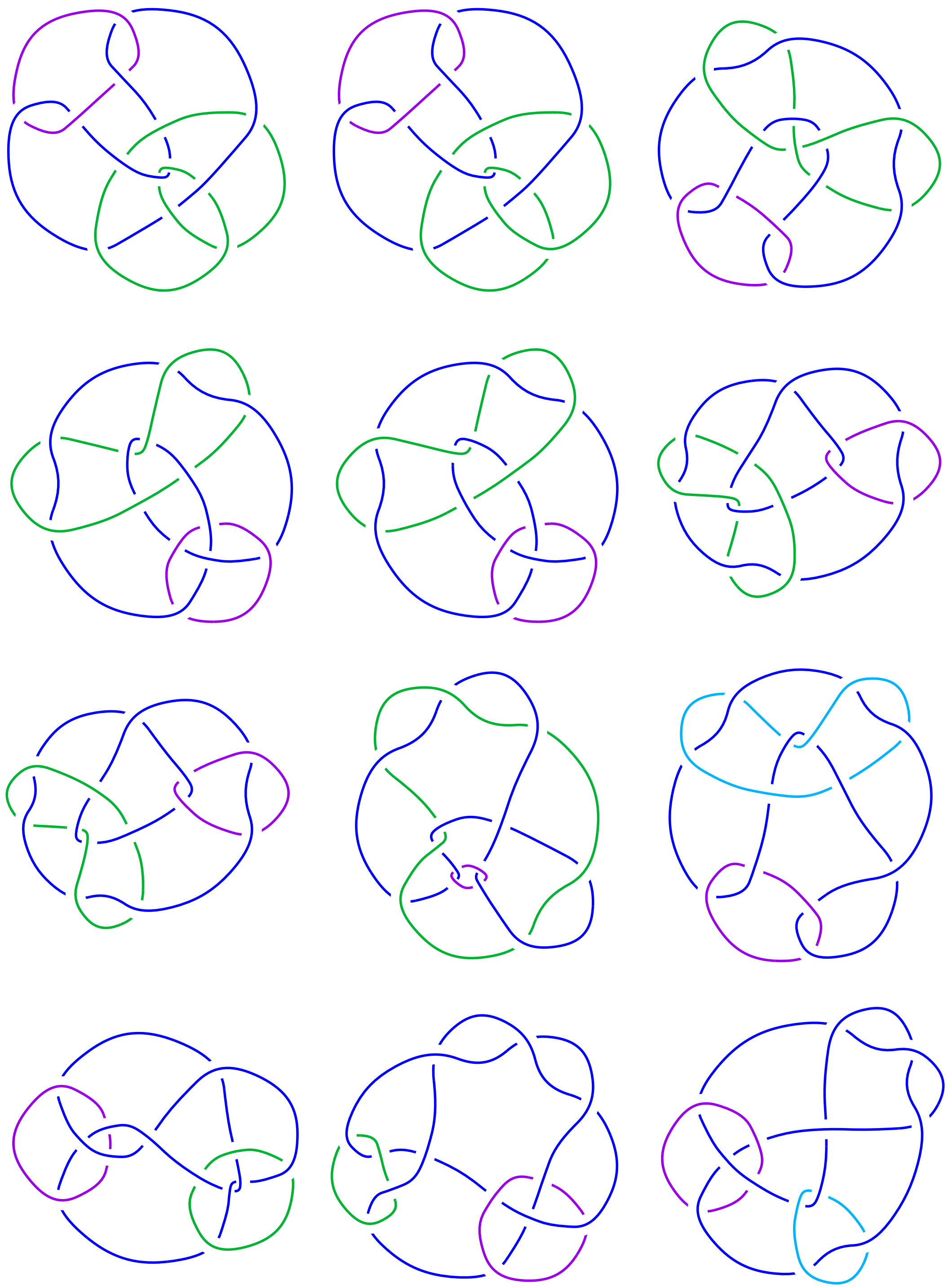


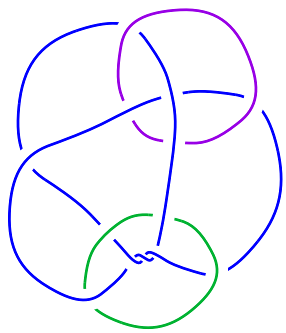
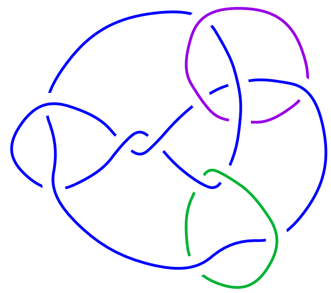
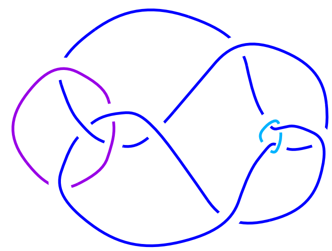
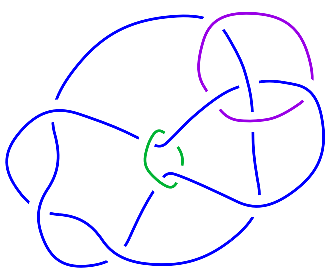
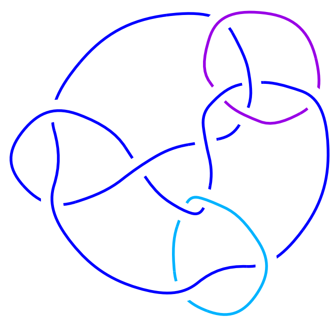
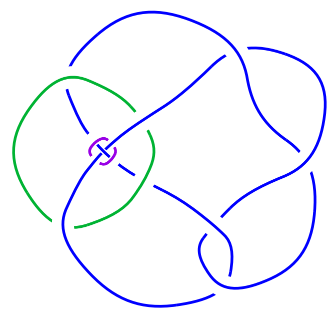
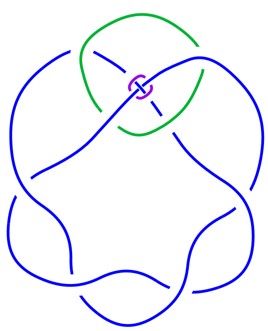
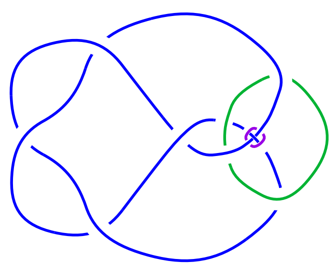
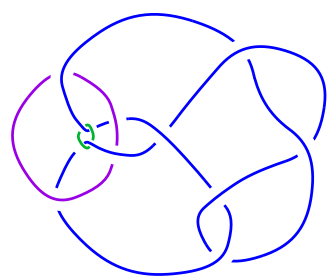
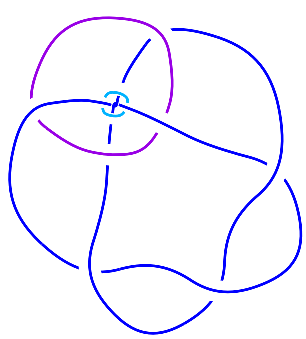
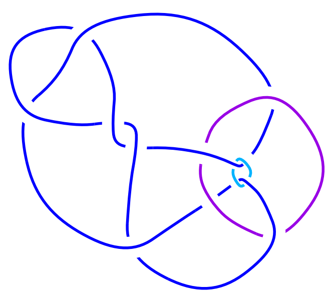
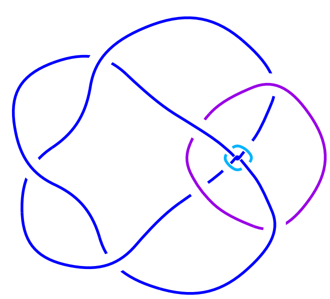


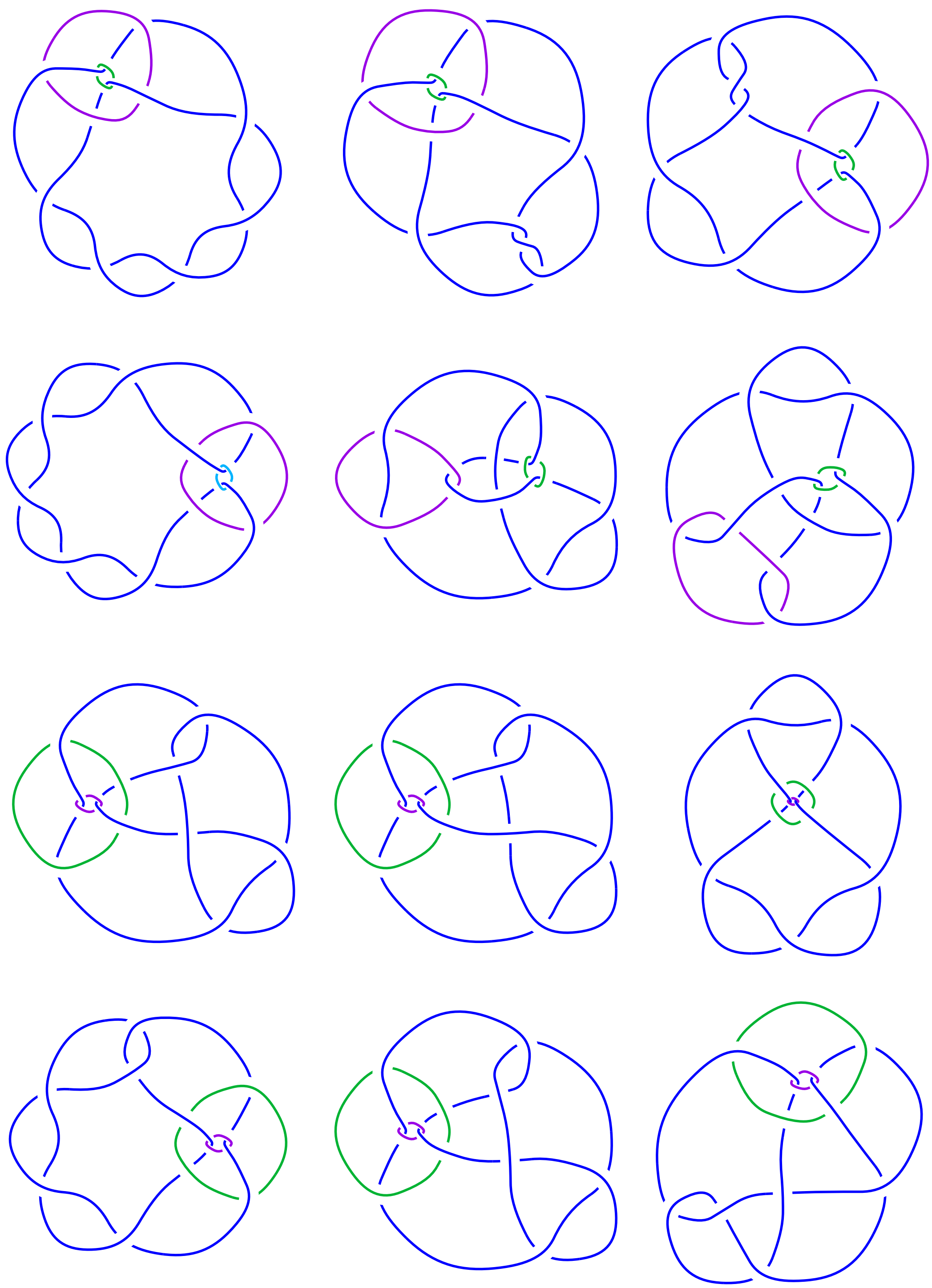













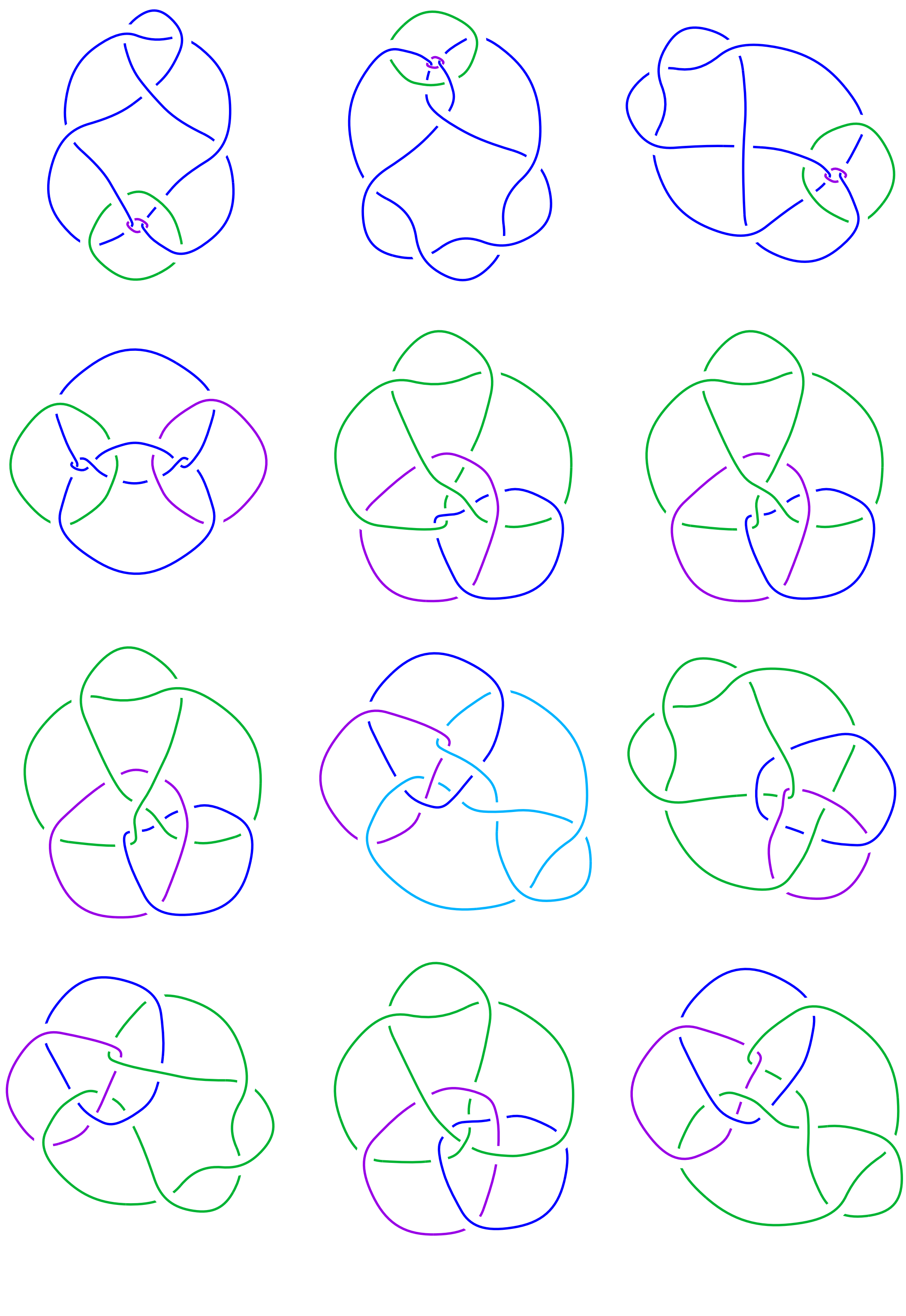


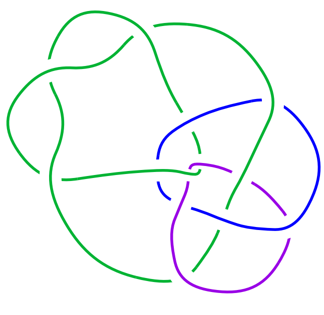
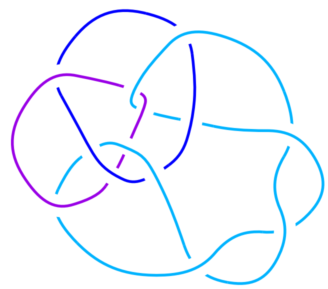
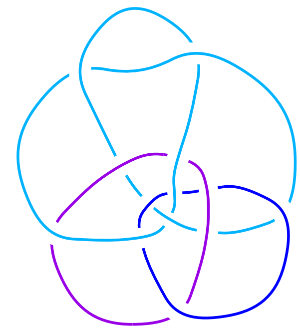
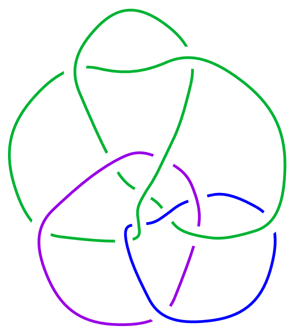
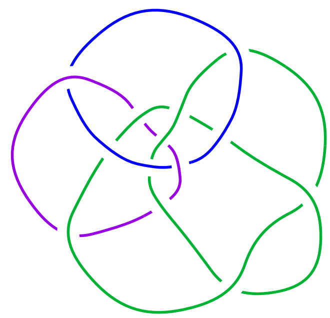
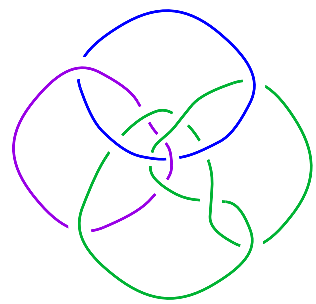
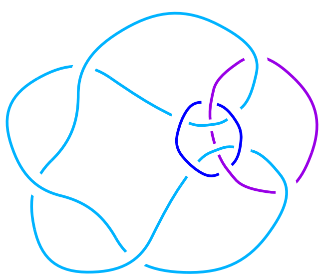
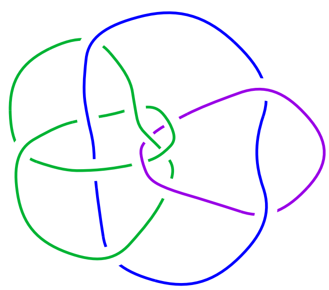
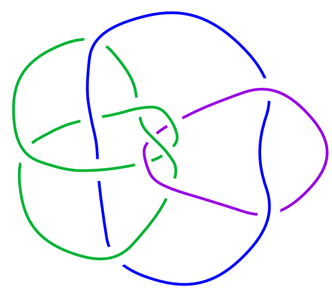
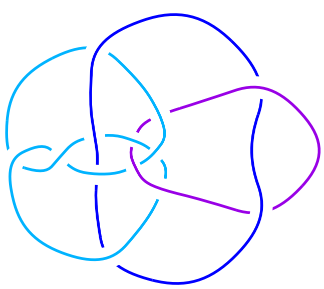
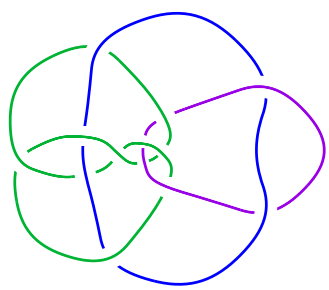
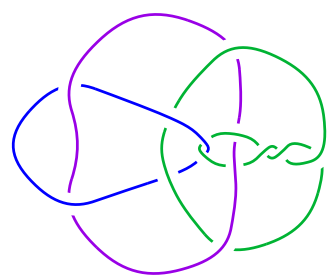


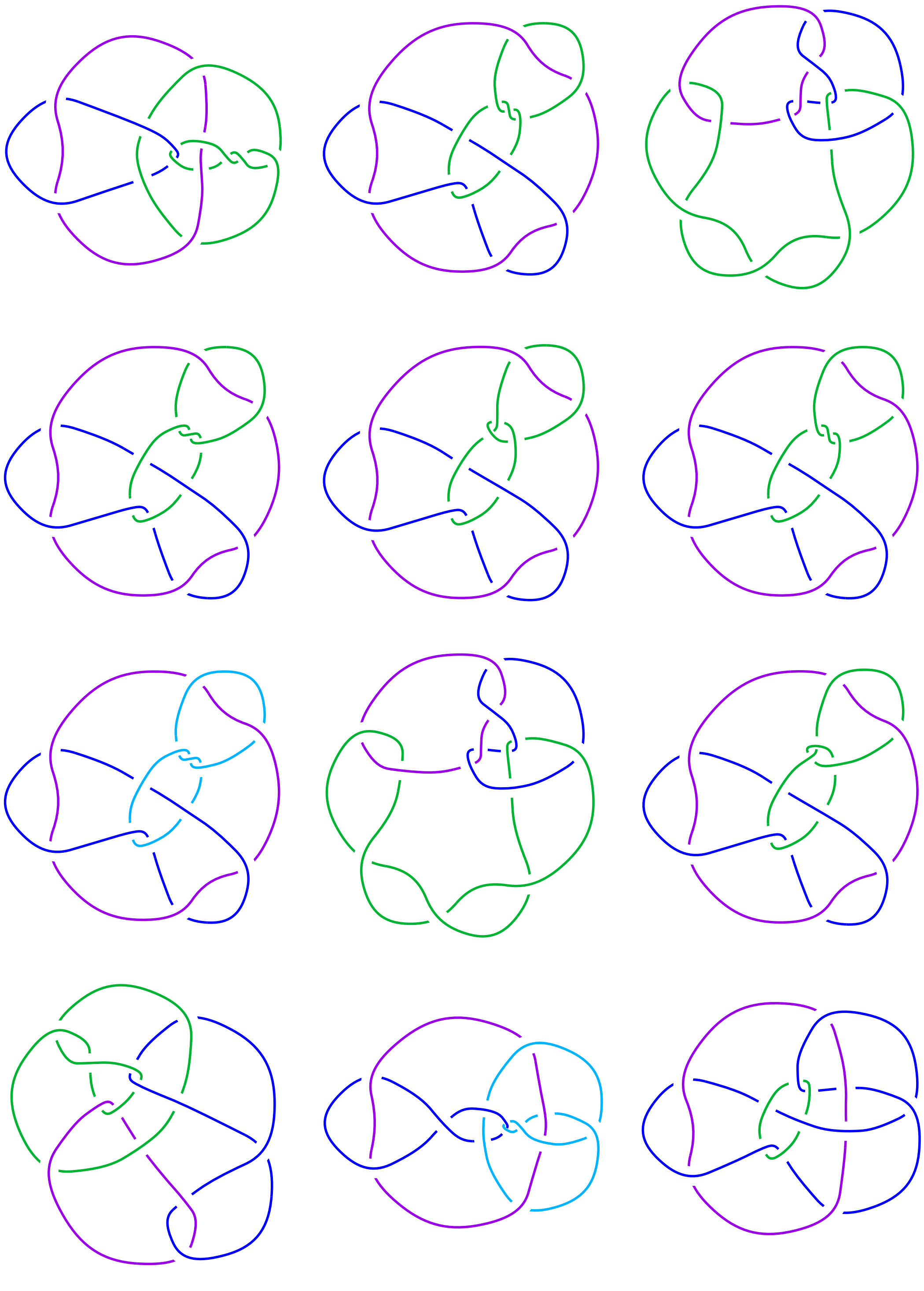


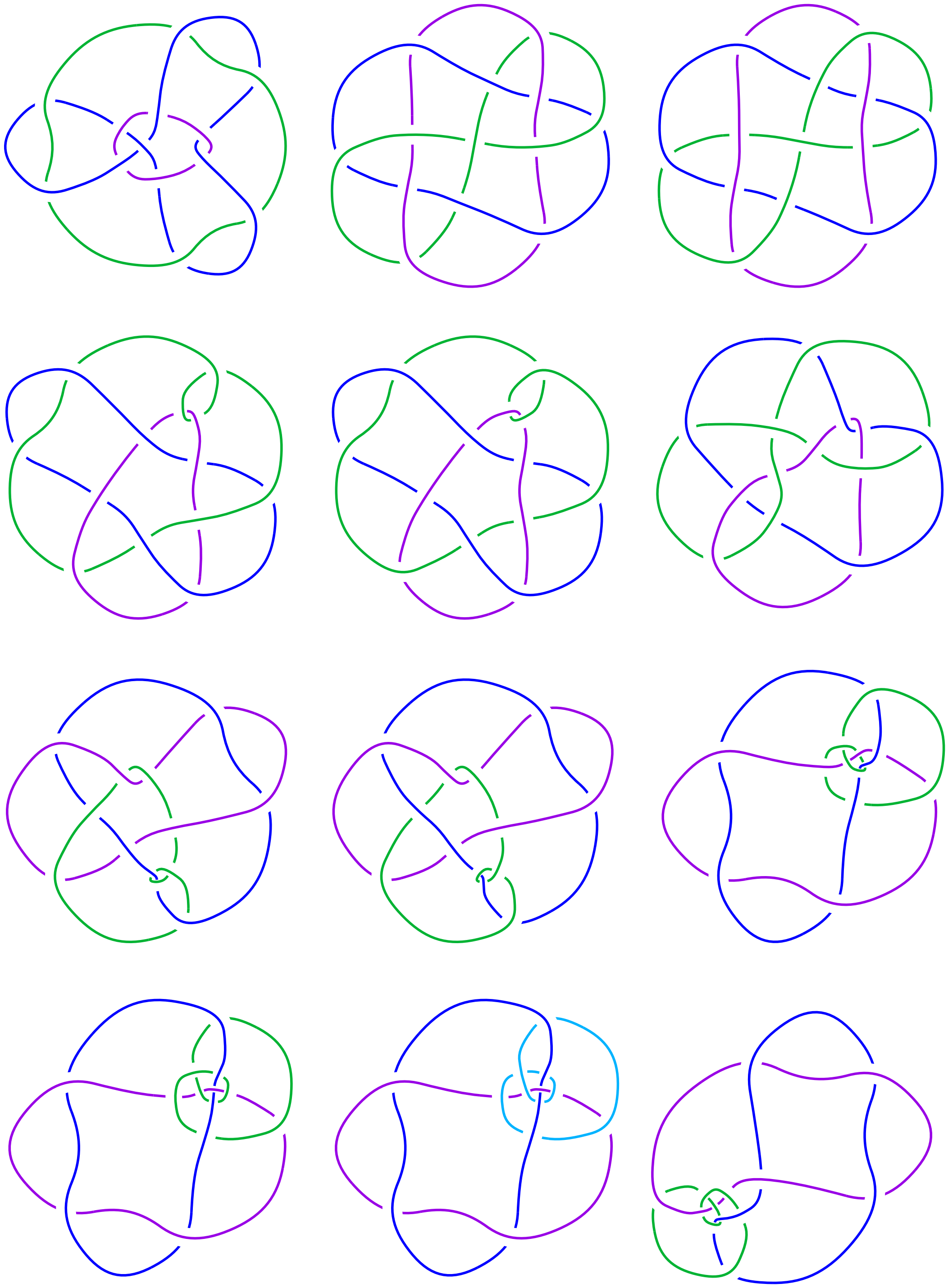













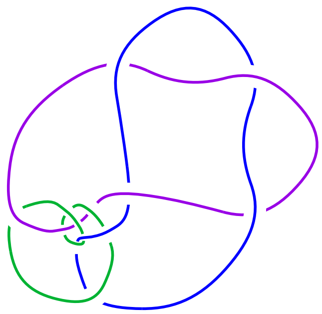
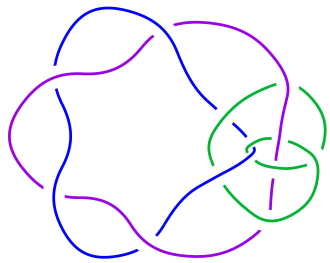
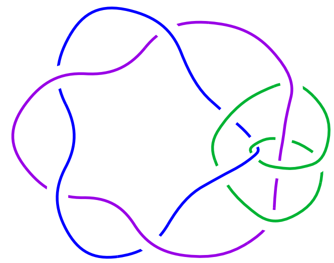
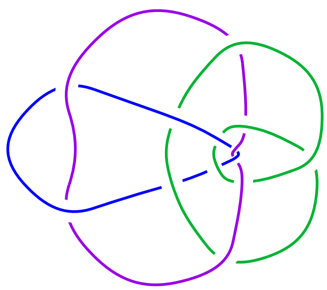
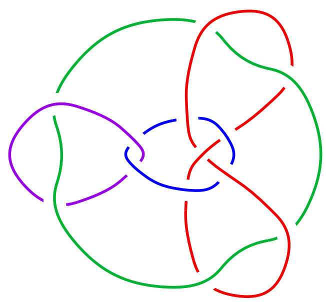
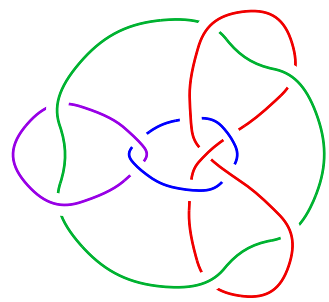
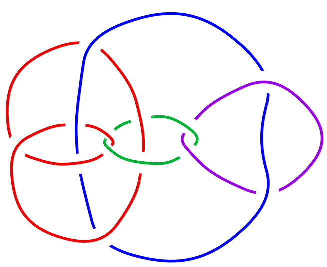
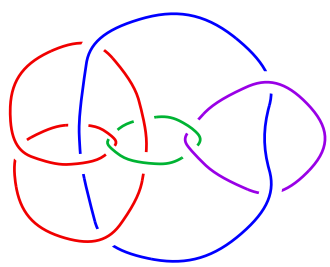
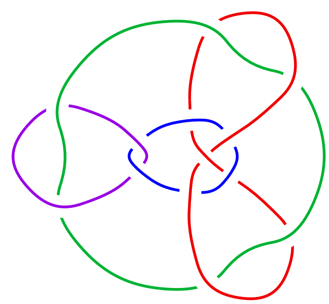
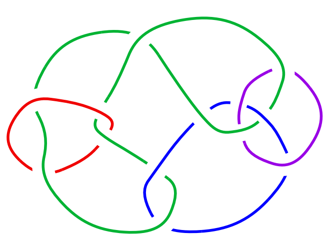
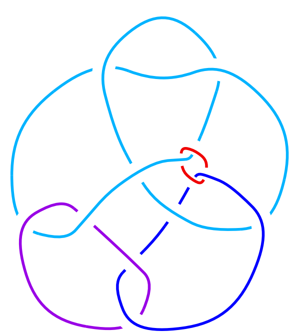
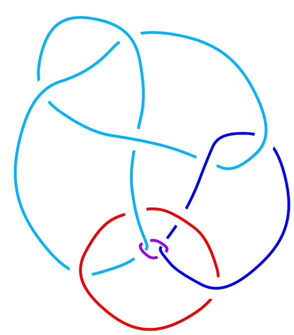


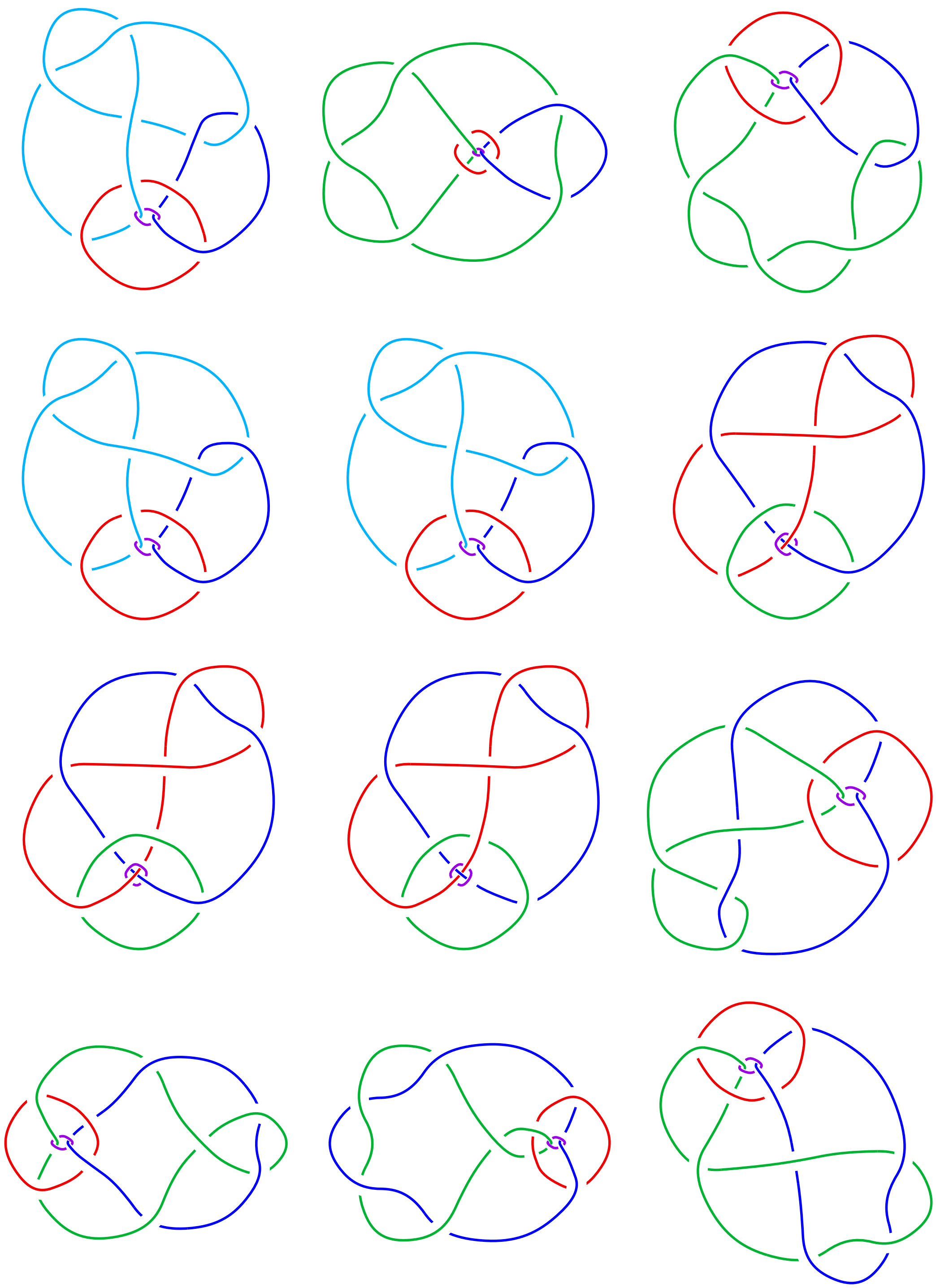


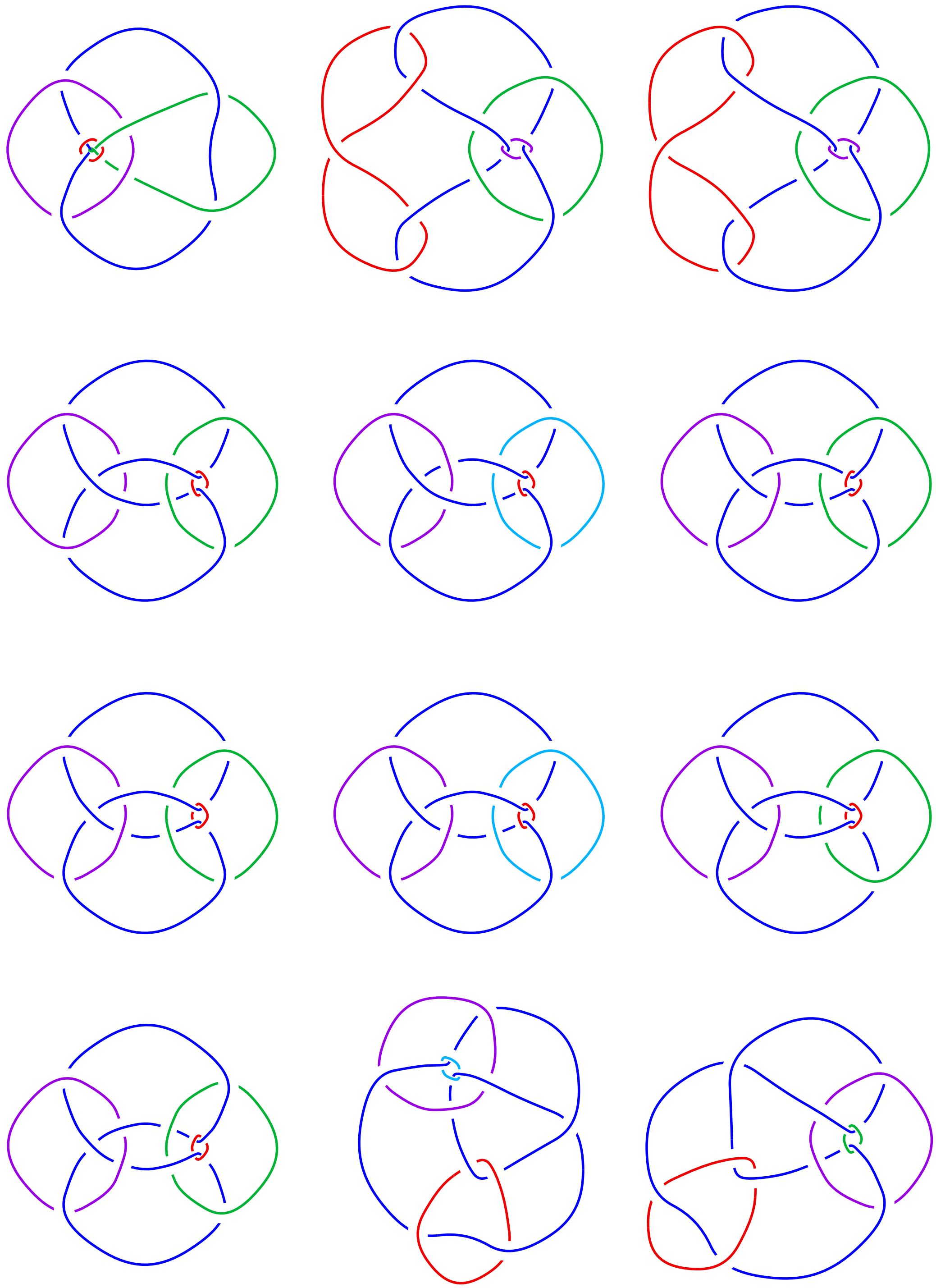













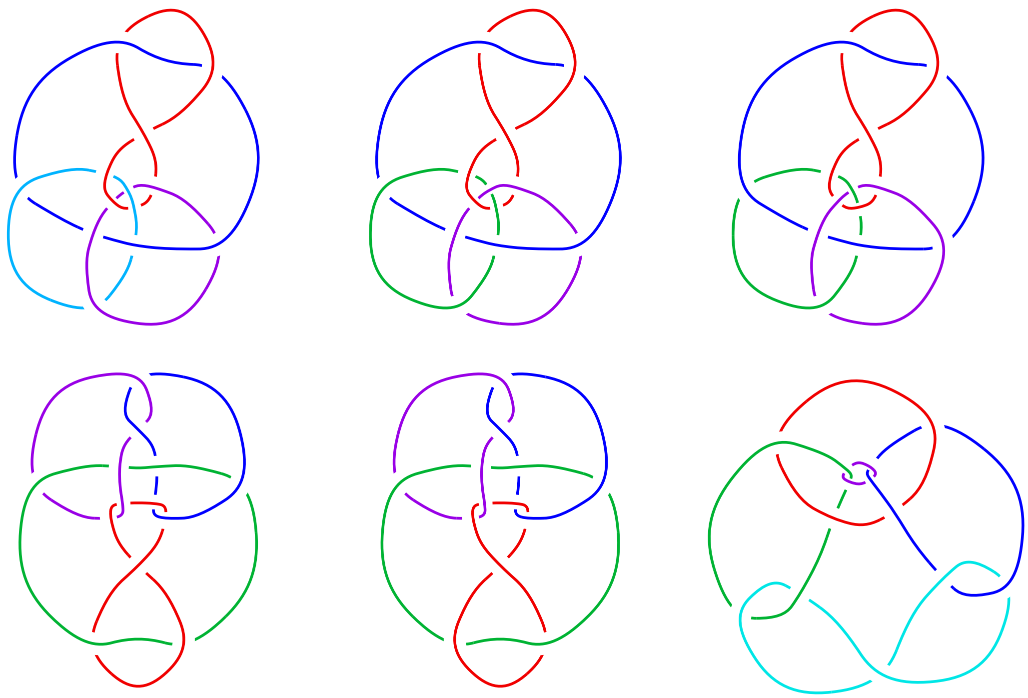










Appendix 3

Alphabetic Codes for the Non-hyperbolic Links

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4	eaecdeab	44	kcbgbceJghKadbFI	84	lcbhbcgaKIHbLEDFJ
5	fbccdefcab	45	kcbgbceJhgKadbFI	85	lcbhbcgfKHajLDbEI
6	fcbbbcDEfAb	46	kcbgbcfaJHbKDGEI	86	lcccfdEGhJkfClaBi
7	gagdefgabc	47	kcccedEGhJifCkBa	87	lcccfdEgHjKFcLAbI
8	gbbecDEfGAb	48	kcccedEghIkbjAcf	88	lcccfdGHjkBEFLcaI
9	hahbdFaGHCE	49	kcccedGHijBFEKac	89	lcccfdgIalJckBEhf
10	hbddefghdabc	50	kcccedgHakIfJEBc	90	lcccfdghalIfkJEbc
11	hcbcccDFgHEAb	51	kcccedgHakIfjEBc	91	lccdedfHaJbKICLGE
12	hcbdbceaGbHDF	52	kccdddfHaIbJCKGE	92	lccdedfhaibJclKGe
13	hdbbbbcEaGBhDf	53	kdbcdbcfaJHbKDGEI	93	lccfcdfhaJcKbLEIG
14	iaiefghiabcd	54	lbdheFgIHkJcLBaD	94	lccfcdfhcJaKbLIEG
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16	ibcfdEfcGhIaB	56	lbdheGlHcjiBDkfa	96	lcdddeFgHiJkLcDaB
17	ibdeeFiGchBDa	57	lbdheHlIaJKDBGFc	97	lcdddeFgidjLkcahB
18	ibdeefiGcaHDb	58	lbdheHlIaKJDBGFc	98	lcdddeFhIcjLKBDaG
19	icbebcdHfIabEG	59	lbdhefHJadIKBLCG	99	ldbccdcFaIKBjlDhEg
20	jajbdGaHIJCEF	60	lbdhefgHdakJlCbi	100	ldbccdcfaIKbjlDhEg
21	jbdfeGjHaIDBFc	61	lbdheglHcakIDbfj	101	ldbccdcfaikbJLdHeG
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23	jbeefghijeabcd	63	lbdheglHdkjICafb	103	ldbcebcfaKHbLIDGEJ
24	jcbddcDGhIJFAbE	64	lbegfDGHIejKAlCB	104	ldbdbdcDGIhKJlAbEF
25	jcbfbcfaIGbJDEH	65	lbegfDHIkCjeLagB	105	ldbdbdcDGIhKjLaBEF
26	jcccddEGhIjBfCa	66	lbegfDgHJIekLABc	106	ldbdbdcdGiHKJLabEF
27	jcccddEgHiJbFcA	67	lbegfDgHKIejLcBA	107	ldbddbcgaKHIbLEDFJ
28	jdbcbccDFHgIjAbE	68	lbegfDgIJceLKABH	108	ldbddbcgaKIHbLEDFJ
29	jdbcbccdFhGIJabE	69	lbegfDgIJcelkABh	109	ldbddbcghKJaLfbDEI
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38	kbdgegkHcajIDbf	78	lcbhbcdEhGIkablfj	118	lebccbbcFaIkBJlEHdg
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Vita

Dylan Joshua Faullin was born in Orange Park, Florida on October 18th, 1979. The son of a sailor, he resided in various locations, eventually settling in Jacksonville, FL, a stone's throw from Orange Park. Here, while attending junior high school, he met his future wife in navy housing. Faullin graduated from Robert E. Lee High School in 1997.

In 2001, Dylan completed his Bachelor of Science with a major in astronomy and minor in mathematics from the University of Florida, Gainesville and moved to Albuquerque, New Mexico to be with his beloved. From 2002 to 2003, he attended the University of New Mexico, concentrating on mathematics.

Dylan entered graduate school at the University of Tennessee, Knoxville in August 2003 and received his Master of Science with a major in mathematics in August 2005.